

New derivation of soliton solutions to the AKNS₂ system via dressing transformation methods

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Abstract

We consider certain boundary conditions supporting soliton solutions in the generalized non-linear Schrödinger equation (AKNS_r) ($r = 1, 2$). Using the dressing transformation (DT) method and the related tau functions we study the AKNS_r system for the vanishing, (constant) non-vanishing and the mixed boundary conditions, and their associated bright, dark, and bright-dark N-soliton solutions, respectively. Moreover, we introduce a modified DT related to the dressing group in order to consider the free field boundary condition and derive generalized N-dark-dark solitons. As a reduced submodel of the AKNS_r system we study the properties of the focusing, defocusing and mixed focusing-defocusing versions of the so-called coupled non-linear Schrödinger equation (r -CNLS), which has recently been considered in many physical applications. We have shown that two-dark-dark-soliton bound states exist in the AKNS₂ system, and three- and higher-dark-dark-soliton bound states can not exist. The AKNS_r ($r \geq 3$) extension is briefly discussed in this approach. The properties and calculations of some matrix elements using level one vertex operators are outlined.

Dedicated to the memory of S. S. Costa.

1 Introduction

Many soliton equations in 1+1 dimensions have integrable multi-component generalizations or more generally integrable matrix generalizations. It is well-known that certain coupled multi-field generalizations of the non-linear Schrödinger equation (CNLS) are integrable and possess soliton type solutions with rich physical properties (see e.g. [1, 2, 3, 4]). The model defined by two coupled NLS systems was earlier studied by Manakov [5]. Another remarkable example of a multi-field generalization of an integrable model is the so-called generalized sine-Gordon model which is integrable in some regions of its parameter space and it has many physical applications [6, 7]. The type of coupled NLS equations find applications in diverse areas of physics such as non-linear optics, optical communication, biophysics, multi-component Bose-Einstein condensate, etc (see e.g. [1, 2, 3, 8, 9]). The multi-soliton solutions of these systems have recently been considered using a variety of methods depending on the initial-boundary values imposed on the solution. For example, the direct methods, mostly the Hirota method, have been applied in [10, 3] and in ref. [11] a wide class of NLS models have been studied in the framework of the Darboux-dressing transformation. Recently, some properties have been investigated, such as the appearance of stationary bound states formed by two dark-dark solitons in the mixed-nonlinearity case (focusing and defocusing) of the 2-CNLS system [4]. The inverse scattering transform method (IST) for the defocusing CNLS model with non-vanishing boundary conditions (NVBC) has been an open problem for over 30 years. The two-component case was solved in [12] and the multi-component model has very recently presented in [13]. The inverse scattering [12, 13] and Hirota's method [10] results on N-dark-dark solitons in the defocusing 2-CNLS model have presented only the degenerate case, i.e. the multi-solitons of the both components are proportional to each other and therefore are reducible to the dark solitons of the scalar NLS model.

The r -CNLS model is related to the AKNS $_r$ system, which is a model with $2r$ dependent variables. As we will show below this system reduces to the r -CNLS model under a particular reality condition. Moreover, general multi-dark-dark soliton solutions of generalized AKNS type systems available in the literature, to our knowledge, are scarce. Recently, there appeared some reports on the general non-degenerate N-dark-dark solitons in the r -CNLS model [4, 14]. In [4] the model with defocusing and mixed nonlinearity is studied in the context of the KP-hierarchy reduction approach. The mixed focusing and defocusing nonlinearity 2-CNLS model presents a two dark-dark soliton stationary bound state; whereas, three or more dark-dark solitons can not form bound states [4]. Here we show that similar phenomena are present in the AKNS $_2$ system. In [14] dark and bright multi-soliton solutions of the r -CNLS model are derived from multi-soliton solutions of the AKNS $_r$ system for arbitrary r in the framework of the algebro-geometric approach.

In this paper we will consider the general constant (vanishing, nonvanishing and mixed vanishing-nonvanishing) boundary value problem for the AKNS $_r$ ($r = 1, 2$) model and show that its particular complexified and reduced version incorporates the focusing and defocusing scalar NLS system for $r = 1$, and the focusing, defocusing and mixed nonlinearity versions of the 2-CNLS system, i.e. the Manakov model,

for $r = 2$, respectively. We will consider the dressing transformation (DT) method to solve integrable non-linear equations, which is based on the Lax pair formulation of the system. In this approach the so-called integrable highest weight representation of the underlying affine Lie algebra is essential to find soliton type solutions (see [15] and references therein). According to the approach of [15] a common feature of integrable hierarchies presenting soliton solutions is the existence of some special “vacuum solutions” such that the Lax operators evaluated on them lie in some Abelian (up to the central term) subalgebra of the associated Kac-Moody algebra. The boundary conditions imposed to the system of equations must be related to the relevant vacuum connections lying in certain Abelian sub-algebra. The soliton type solutions are constructed out of those “vacuum solutions” through the so called soliton specialization in the context of the DT. These developments lead to a quite general definition of tau functions associated to the hierarchies, in terms of the so called “integrable highest weight representations” of the relevant Kac-Moody algebra. However, the free field boundary condition does not allow the vacuum connections to lie in an Abelian sub-algebra; so, we will introduce a modified DT in order to deal with this case. We consider a modified DT relying on the dressing group composition law of two successive DT’s [29]. The corresponding tau functions will provide generalized dark-dark soliton solutions possessing additional parameters as compared to the ones obtained with constant boundary conditions.

Here we adopt a hybrid of the DT and Hirota methods [7, 18] to obtain soliton solutions of the AKNS system. As pointed out in [17], the Hirota method presents some drawbacks in order to derive a general type of soliton solutions, noticeably it is not appropriate to handle vector NLS type models in a general form since the process relies on a insightful guess of the functional forms of each component, whereas in the group theoretical approach adopted here the dependence of each component on the generalized tau functions become dictated by the DT and the solutions in the orbit of certain vacuum solutions are constructed in a uniform way. The generalized tau-functions for the nonlinear systems are defined as an alternative set of variables corresponding to certain matrix elements evaluated in the integrable highest-weight representations of the underlying affine Kac-Moody algebra. In this way we overcome two difficulties of the methods. The Hirota method needs the tau-functions and an expansion for them, but it does not provide a recipe how to construct them. That is accurately solved through the DT method, which in its turn needs the evaluation of certain matrix elements in the vertex operator representation. We may avoid the cumbersome matrix element calculations through the Hirota expansion method. Since this method is recursive it allows a simple implementation on a computer program for algebraic manipulation like MATHEMATICA.

The paper is organized as follows. In section 2 we present the generalized non-linear Schrödinger equation (AKNS₂) associated to the homogeneous gradation of the Kac-Moody algebra \hat{sl}_3 . The 2–component coupled non-linear Schrödinger equation (2–CNLS) is defined as a particular reduction by imposing certain conditions on the AKNS₂ fields. In section 3 we review the theory of the DT, describe the various constant boundary conditions and define the tau-functions. In 3.1 we present the modified DT associated to the dressing group suitable for free field boundary conditions. In section 4 we apply the DT method to the vanishing boundary

conditions and derive the bright solitons. In section 5 we consider the non-vanishing boundary conditions and derive the AKNS_r ($r = 1, 2$) dark solitons. Moreover, the general N-dark-dark solitons with free field NVBC of the AKNS₂ are derived through the modified DT method. In 5.3 we discuss the dark-dark soliton bound states. In 6 the dark-bright solitons are derived associated to the mixed boundary conditions. In the section 7 we briefly discuss the AKNS_r ($r \geq 3$) extension in the framework of the DT methods, and in the section 8 we discuss the main results of the paper. Finally, we have included the $\hat{sl}(2)$ and $\hat{sl}(3)$ affine Kac-Moody algebra properties, in appendices A and B, respectively. In C the matrix elements in $\hat{sl}(3)$ have been computed using vertex operator representations.

2 The model

The AKNS₂ model can be constructed in the framework of the Zakharov-Shabat formalism using the hermitian symmetric spaces [19]. In [20] an affine Lie algebraic formulation based on the loop algebra $g \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ of $g \in sl(3)$ has been presented. Here instead we construct the model associated to the full affine Kac-Moody algebra $\mathcal{G} = \widehat{sl}(3)$ with homogeneous gradation and a semi-simple element $E^{(l)}$ (see appendix B). So, the connections are given by

$$A = E^{(1)} + \sum_{i=1}^2 \Psi_i^+ E_{\beta_i}^{(0)} + \sum_{i=1}^2 \Psi_i^- E_{-\beta_i}^{(0)} + \phi_1 C, \quad (2.1)$$

$$\begin{aligned} B = & E^{(2)} + \sum_{i=1}^2 \Psi_i^+ E_{\beta_i}^{(1)} + \sum_{i=1}^2 \Psi_i^- E_{-\beta_i}^{(1)} + \sum_{i=1}^r \partial_x \Psi_i^+ E_{\beta_i}^{(0)} - \sum_{i=1}^r \partial_x \Psi_i^- E_{-\beta_i}^{(0)} - \\ & [\Psi_1^+ \Psi_1^- - \Omega_1](H_1^{(0)} + H_2^{(0)}) - [\Psi_2^+ \Psi_2^- - \Omega_2]H_2^{(0)} - \\ & \Psi_1^+ \Psi_2^- E_{\beta_1 - \beta_2}^{(0)} - \Psi_2^+ \Psi_1^- E_{\beta_2 - \beta_1}^{(0)} + \phi_2 C, \end{aligned} \quad (2.2)$$

where Ψ_i^+ , Ψ_i^- , ϕ_1 and ϕ_2 are the fields of the model. In this construction we consider the fields as being real, however some reductions and complexification procedures will be performed below. Notice that the auxiliary fields ϕ_1 and ϕ_2 lie in the direction of the affine Lie algebra central term C . The existence of this term plays an important role in the theory of the so-called integrable highest weight representations of the Kac-Moody algebra and they will be important below in order to find the soliton type solutions of the model. The connections (2.1)-(2.2) conveniently incorporate the terms with the constant parameters $\Omega_{1,2}$ in order to take into account the various boundary conditions, so, these differ slightly from the ones in [20, 21]. The $\hat{sl}(3)$ Kac-Moody algebra conventions and notations are presented in appendix B. In the basis considered in this paper one has

$$E^{(l)} = \frac{1}{3} \sum_{a=1}^2 a H_a^{(l)} \quad (2.3)$$

The β_i in (2.1)-(2.2) are positive roots defined in (B.24) such that

$$\beta_1 \equiv \alpha_1 + \alpha_2; \quad \beta_2 \equiv \alpha_2, \quad \alpha_1, \alpha_2 = \text{simple roots} \quad (2.4)$$

Notice that the connections lie in the subspaces defined in (B.12)

$$A \in \widehat{g}_o + \widehat{g}_1, \quad B \in \widehat{g}_o + \widehat{g}_1 + \widehat{g}_2, \quad [D, \widehat{g}_n] = n \widehat{g}_n \quad (2.5)$$

The zero curvature condition $[\partial_t - B, \partial_x - A] = 0$ supplied with the $\widehat{sl}(3)$ commutation relations (B.4)-(B.11) and (B.25)-(B.31) provides the following system of equations

$$\partial_t \Psi_i^+ = +\partial_x^2 \Psi_i^+ - 2 \left[\sum_{j=1}^2 \Psi_j^+ \Psi_j^- - \frac{1}{2} (\beta_i \cdot \vec{\Omega}) \right] \Psi_i^+, \quad i = 1, 2 \quad (2.6)$$

$$\partial_t \Psi_i^- = -\partial_x^2 \Psi_i^- + 2 \left[\sum_{j=1}^2 \Psi_j^+ \Psi_j^- - \frac{1}{2} (\beta_i \cdot \vec{\Omega}) \right] \Psi_i^-, \quad (2.7)$$

$$\partial_t \bar{\phi}_1 - \partial_x \bar{\phi}_2 = 0; \quad (2.8)$$

$$\vec{\Omega} \equiv \sum_{i=1}^2 \Omega_i \beta_i, \quad (\beta_1 \cdot \vec{\Omega}) = 2\Omega_1 + \Omega_2, \quad (\beta_2 \cdot \vec{\Omega}) = 2\Omega_2 + \Omega_1, \quad \Omega_{1,2} = \text{const.} \quad (2.9)$$

Notice that the auxiliary fields $\phi_{1,2}$ completely decouple from the AKNS₂ fields Ψ_j^\pm . The constant parameters $\Omega_{1,2}$ will be related below to certain boundary conditions and some trivial solutions of the system (2.6)-(2.7).

The integrability of the system of equations (2.6)-(2.7), for $\Omega_{1,2} = 0$, and its multi-Hamiltonian structure have been established [21]. A version of (2.6)-(2.7) for arbitrary r has recently been considered in [14]. The system of equations (2.6)-(2.7) supplied with a convenient complexification can be related to some versions of the so-called coupled non-linear Schrödinger equation (CNLS) [5, 1, 2, 3]. For example by making

$$t \rightarrow -it, \quad [\Psi_i^+]^* = -\mu \delta_i \Psi_i^- \equiv -\mu \delta_i \psi_i, \quad (2.10)$$

where \star means complex conjugation, $\mu \in \mathbb{R}_+$, $\delta_i = \pm 1$, we may reduce the system (2.6)-(2.7) to the well known 2-coupled non-linear Schrödinger system (2-CNLS)

$$i \partial_t \psi_k + \partial_x^2 \psi_k + 2\mu \left(\sum_{j=1}^2 \delta_j |\psi_j|^2 - \frac{1}{2} |(\beta_k \cdot \vec{\Omega})| \right) \psi_k = 0, \quad k = 1, 2. \quad (2.11)$$

The term $(\beta_k \cdot \vec{\Omega})$ is provided in (2.9) and the parameter $\mu > 0$ represents the strength of nonlinearity and the coefficients δ_j define the sign of the nonlinearity. The system (2.11) can be classified into three classes depending on the signs of the nonlinearity coefficients δ_i . For $\delta_1 = \delta_2 = 1$ this system is the focusing Manakov model which supports bright-bright solitons [5]. For $\delta_1 = \delta_2 = -1$, it is the defocusing Manakov model which supports bright-dark and dark-dark solitons [10, 12, 22]. In the cases $\delta_1 = -\delta_2 = \pm 1$ one has the mixed focusing-defocusing nonlinearities. In this case, these equations support bright-bright solitons [23, 3], bright-dark solitons [24]. The defocusing and mixed nonlinearity cases have recently been considered in [4] through the KP-hierarchy reduction method and dark-dark solitons have been obtained. The system (2.11) has also been considered in the study of oscillations and interactions of dark and dark-bright solitons in Bose-Einstein condensates [25, 26]. The multi-dark-dark solitons in the mixed non-linearity case are useful

for many physical applications such as nonlinear optics, water waves and Bose-Einstein condensates, where the generally coupled NLS equations often appear.

The focusing CNLS system possesses a remarkable type of soliton solution undergoing a shape changing (inelastic) collision property due to intensity redistribution among its modes. In this context, it has been found a novel class of solutions called partially coherent solitons (PCS) which are of substantially variable shape, such that under collisions the profiles remain stationary [1, 27, 28]. Interestingly, the PCSs, namely, 2-PCS, 3-PCS, ..., r -PCS, are special cases of the well known 2-soliton, 3-soliton, ..., r -soliton solutions of the 2-CNLS, 3-CNLS, ..., r -CNLS equations, respectively [2, 3]. So, the understanding of the variable shape collisions and many other properties of these partially coherent solitons can be studied by providing the higher-order soliton solutions of the r -CNLS ($r \geq 2$) system considered as submodel of the relevant AKNS $_r$. We believe that the group theoretical point of view of finding the analytical results for the general case of N -soliton interactions would facilitate the study of their properties; for example, the asymptotic behavior of trains of N solitonlike pulses with approximately equal amplitudes and velocities, as studied in [16]. Notice that the set of solutions of the AKNS model (2.6)-(2.7) is much larger than the solutions of the CNLS system (2.11), since only the solutions of the former which satisfy the constraints (2.10) will be solutions of the CNLS model (2.11). This fact will be seen below in many instances; so, we believe that the AKNS $_r$ soliton properties with relevant boundary conditions deserve a further study.

We will show that the three classes mentioned above, i.e. the *focusing*, *defocusing* and the *mixed focusing-defocusing* CNLS model can be related respectively to the vanishing, nonvanishing and mixed vanishing-nonvanishing boundary conditions of the AKNS $_r$ model in the framework of the DT approach adapted conveniently to each case. The (constant) non-vanishing boundary conditions require the extension of the DT method to incorporate non-zero constant vacuum solutions. Therefore, the vertex operators corresponding to the vanishing boundary case undergo a generalization in such a way that the nilpotency property, which is necessary in obtaining soliton solutions, should be maintained. Using the modified vertex operators we construct multi-soliton solutions in the cases of (constant) non-vanishing and *mixed* vanishing-nonvanishing boundary conditions. The free field NVBC requires a modification of the usual DT of [15] by considering two successive DT's in the context of the dressing group [29]. However, the same vertex operator generating the dark-dark solitons in the (constant) NVBC will be used in the free field NVBC case with a modified tau functions.

3 Dressing transformations for AKNS $_2$

In this section we summarize the so-called DT procedure to find soliton solutions, which works by choosing a vacuum solution and then mapping it into a non-trivial solution, following the approach of [15]. For simplicity, we concentrate on a version of the DT suitable for vanishing, (constant) non-vanishing and mixed boundary conditions of the AKNS $_2$ model (2.6)-(2.8). The free field boundary condition requires a modified

DT which is developed in subsection 3.1. So, let us consider

$$\lim_{x \rightarrow -\infty} \Psi_j^\pm \rightarrow \rho_{j,L}^\pm; \quad \lim_{x \rightarrow +\infty} \Psi_j^\pm \rightarrow \rho_{j,R}^\pm; \quad \phi_{1,2} \rightarrow 0; \quad \rho_{j,L}^\pm, \rho_{j,R}^\pm = \text{const.} \quad (3.1)$$

We may identify the NVBC (3.1) to certain classes of trivial vacuum solutions of the system (2.6)-(2.7):

1) the trivial zero vacuum solution

$$\Psi_{j,vac}^\pm = \rho_j^\pm = 0, \quad j = 1, 2 \quad (3.2)$$

2) the trivial constant vacuum solution

$$\Psi_{j,vac}^\pm = \rho_j^\pm \neq 0, \quad j = 1, 2. \quad (3.3)$$

Notice that (3.3) is a trivial constant vacuum solution of (2.6)-(2.7) provided that the expression $[\sum_{j=1}^2 \rho_j^+ \rho_j^- - \frac{1}{2}(\beta_i \cdot \vec{\Omega})]$ vanishes for any $i = 1, 2$; which is achieved if $\Omega_1 = \Omega_2 \equiv \Omega$, implying $\sum_{j=1}^2 \rho_j^+ \rho_j^- = \frac{3}{2}\Omega$.

3) the mixed constant-zero (zero-constant) vacuum solutions

$$i) \Psi_{1,vac}^\pm = \rho_1^\pm \neq 0, \quad \Psi_{2,vac}^\pm = \rho_2^\pm = 0, \quad (3.4)$$

$$ii) \Psi_{1,vac}^\pm = \rho_1^\pm = 0, \quad \Psi_{2,vac}^\pm = \rho_2^\pm \neq 0. \quad (3.5)$$

The first mixed trivial solution (3.4) requires $2\rho_1^+ \rho_1^- = 2\Omega_1 + \Omega_2$, whereas for the second trivial solution (3.5) it must be $2\rho_2^+ \rho_2^- = 2\Omega_2 + \Omega_1$.

The connections (2.1)-(2.2) for the above vacuum solution (3.3) take the form

$$A^{vac} \equiv E^{(1)} + \rho_1^+ E_{\beta_1}^{(0)} + \rho_1^- E_{-\beta_1}^{(0)} + \rho_2^+ E_{\beta_2}^{(0)} + \rho_2^- E_{-\beta_2}^{(0)}, \quad (3.6)$$

$$B^{vac} \equiv E^{(2)} + \rho_1^+ E_{\beta_1}^{(1)} + \rho_1^- E_{-\beta_1}^{(1)} + \rho_2^+ E_{\beta_2}^{(1)} + \rho_2^- E_{-\beta_2}^{(1)} - \rho_1^+ \rho_2^- E_{\beta_1 - \beta_2}^{(0)} - \rho_2^+ \rho_1^- E_{\beta_2 - \beta_1}^{(0)} - (\rho_1^+ \rho_1^- - \Omega_1)(H_1^{(0)} + H_2^{(0)}) - (\rho_2^+ \rho_2^- - \Omega_2)H_2^{(0)}. \quad (3.7)$$

Notice that $[A^{vac}, B^{vac}] = 0$ and in order to define these vacuum connections it suffices to consider the constant values of $\rho_{j,L(R)}^\pm$ in (3.1) related to one of the limits $x \rightarrow \pm\infty$, say $\rho_{j,L}^\pm \equiv \rho_j^\pm$ as above. These connections are related to the group element $\Psi^{(0)}$ through

$$A^{(vac)} = \partial_x \Psi^{(0)} [\Psi^{(0)}]^{-1}; \quad B^{(vac)} = \partial_t \Psi^{(0)} [\Psi^{(0)}]^{-1}, \quad (3.8)$$

where

$$\Psi^{(0)} = e^{xA^{vac} + tB^{vac}}. \quad (3.9)$$

The DT is implemented through two gauge transformations generated by Θ_\pm such that the nontrivial gauge connections in the vacuum orbit become [15]

$$A = \Theta_\pm^h A^{(vac)} [\Theta_\pm^h]^{-1} + \partial_x \Theta_\pm^h [\Theta_\pm^h]^{-1} \quad (3.10)$$

$$B = \Theta_\pm^h B^{(vac)} [\Theta_\pm^h]^{-1} + \partial_t \Theta_\pm^h [\Theta_\pm^h]^{-1} \quad (3.11)$$

where

$$\Theta_-^h = \exp \left(\sum_{n>0} \sigma_{-n} \right), \quad \Theta_+^h \equiv M^{-1}N; \quad M = \exp(\sigma_o), \quad N = \exp \left(\sum_{n>0} \sigma_n \right), \quad (3.12)$$

where $[D, \sigma_n] = n \sigma_n$. Therefore, from the relationships

$$A = \partial_x(\Psi^h)[\Psi^h]^{-1}; \quad B = \partial_t(\Psi^h)[\Psi^h]^{-1}, \quad \Psi^h \equiv \Theta_{\pm}^h \Psi^{(0)}, \quad (3.13)$$

one has

$$[\Theta_-^h]^{-1} \Theta_+^h = \Psi^{(0)} h [\Psi^{(0)}]^{-1}, \quad (3.14)$$

where h is some constant group element.

One can relate the fields Ψ_i^{\pm} , ϕ_1 and ϕ_2 to some of the components in σ_n . One has

$$A = A^{vac} + [\sigma_{-1}, E^{(1)}] + \text{terms of negative grade.} \quad (3.15)$$

$$\begin{aligned} B = & B^{vac} + [\sigma_{-1}, E^{(2)}] + [\sigma_{-2}, E^{(2)}] + \frac{1}{2} [\sigma_{-1}, [\sigma_{-1}, E^{(2)}]] + \left[\sigma_{-1}, \sum_{i=1}^2 \rho_i^+ E_{\beta_i}^{(1)} \right] + \\ & \left[\sigma_{-1}, \sum_{i=1}^2 \rho_i^- E_{-\beta_i}^{(1)} \right] + \text{terms of negative grade} \end{aligned} \quad (3.16)$$

Taking into account the grading structure of the connection A in (2.5) we may write the σ'_n s in terms of the fields of the model. In order to match the zero grade terms of the both sides of the equation (3.15) one must have

$$\sigma_{-1} = - \sum_{i=1}^2 (\Psi_i^+ - \rho_i^+) E_{\beta_i}^{(-1)} + \sum_{i=1}^2 (\Psi_i^- - \rho_i^-) E_{-\beta_i}^{(-1)} + \sum_{a=1}^2 \sigma_{-1}^a H_a^{(-1)}. \quad (3.17)$$

In the equation above, the explicit form of σ_{-1}^{\pm} in terms of the fields Ψ_i^{\pm} can be obtained by setting the sum of the (-1) grade terms to zero. Nevertheless, the form of the σ_{-1}^a will not be necessary for our purposes.

Following the above procedure to match the gradations on both sides of eqs. (3.15)-(3.16) one notices that the σ_{-n} 's with $n \geq 1$ are used to cancel out the undesired components on the r.h.s. of the equations.

From the equations (3.12)-(3.13) and (3.14) one has

$$\langle \lambda | M^{-1} | \lambda' \rangle = \langle \lambda | [\Psi^{(0)} h \Psi^{(0)-1}] | \lambda' \rangle, \quad (3.18)$$

where $\langle \lambda |$ and $| \lambda' \rangle$ are certain states annihilated by $\mathcal{G}_{<}$ and $\mathcal{G}_{>}$, respectively. Defining

$$\sigma_o = \sum_{\alpha>0} \sigma_o^{\alpha} E_{\alpha}^{(0)} + \sum_{\alpha>0} \sigma_o^{-\alpha} E_{-\alpha}^{(0)} + \sum_{a=1}^2 \sigma_o^a \mathbf{H}_a^{(0)} + \eta C \quad (3.19)$$

and choosing a specific matrix element one gets a space time dependence for the field η

$$e^{-\eta} = \langle \lambda_0 | [\Psi^{(0)} h \Psi^{(0)-1}] | \lambda_0 \rangle \quad (3.20)$$

$$\equiv \tau_0, \quad (3.21)$$

where we have defined the tau function τ_0 and $|\lambda_0\rangle$ is the highest weight defined in (B.15). Next, we will write the fields Ψ_i^\pm in terms of certain matrix elements. These functions will be represented as matrix elements in an appropriate representation of the affine Lie algebra $\widehat{sl}(3)$. We proceed by writing the eq. (3.14) in the form

$$\exp\left(-\sum_{n>0}\sigma_{-n}\right)|\lambda_o\rangle = \left[\Psi^{(0)}h\Psi^{(0)-1}\right]|\lambda_o\rangle \tau_0^{-1}, \quad (3.22)$$

where the eqs. (3.12), (3.19) and (3.20)-(3.21) have been used.

Then the terms with grade (-1) in the both sides of (3.22) can be written as

$$-\sigma_{-1}|\lambda_o\rangle = \frac{\left[\Psi^{(0)}h\Psi^{(0)-1}\right]_{(-1)}|\lambda_o\rangle}{\tau_0(x,t)} \quad (3.23)$$

or equivalently

$$\left(\sum_{i=1}^2(\Psi_i^+ - \rho_i^+)E_{\beta_i}^{(-1)} - \sum_{i=1}^2(\Psi_i^- - \rho_i^-)E_{-\beta_i}^{(-1)} - \sum_{a=1}^2\sigma_{-1}^a H_a^{(-1)}\right)|\lambda_o\rangle = \frac{\left[\Psi^{(0)}h\Psi^{(0)-1}\right]_{(-1)}|\lambda_o\rangle}{\tau_0(x,t)}. \quad (3.24)$$

Acting on the left in eq. (3.24) by $E_{\pm\beta_i}^{(1)}$ and taking the relevant matrix element with the dual highest weight state $\langle\lambda_o|$ we may have

$$\Psi_i^+ = \rho_i^+ + \frac{\tau_i^+}{\tau_0} \quad \text{and} \quad \Psi_i^- = \rho_i^- - \frac{\tau_i^-}{\tau_0}; \quad i = 1, 2. \quad (3.25)$$

where the *tau* functions τ_i^\pm , τ_0 are defined by

$$\tau_i^+(x,t) \equiv \langle\lambda_o| E_{-\beta_i}^{(1)} \left[\Psi^{(0)}h\Psi^{(0)-1}\right]_{(-1)}|\lambda_o\rangle, \quad (3.26)$$

$$\tau_i^-(x,t) \equiv \langle\lambda_o| E_{\beta_i}^{(1)} \left[\Psi^{(0)}h\Psi^{(0)-1}\right]_{(-1)}|\lambda_o\rangle, \quad (3.27)$$

$$\tau_0(x,t) \equiv \langle\lambda_o| \left[\Psi^{(0)}h\Psi^{(0)-1}\right]_{(o)}|\lambda_o\rangle. \quad (3.28)$$

Notice that in order to get the above relationships we have used the commutation rules for the corresponding $\widehat{sl}(3)$ affine Kac-Moody algebra elements (B.4)-(B.11) and (B.25)-(B.31), as well as their properties acting on the highest weight state $|\lambda_o\rangle$ (B.15).

According to the solitonic specialization in the context of the DT method the soliton solutions are determined by choosing suitable constant group elements h in the eqs. (3.26)-(3.28). In order to obtain N -soliton solutions the general prescription is to parameterize the orbit of the vacuum as a product of exponentials of eigenvectors of the operators ε_l ($\varepsilon_1 = A^{vac}$, $\varepsilon_2 = B^{vac}$) defined in (3.6)-(3.7); i.e $h = \prod_{i=1}^N e^{F_i}$, where $[\varepsilon_l, F_i] = \lambda_i^l F_i$, such that $(F_i)^m \neq 0$ only for $m < m_i$, m_i being some positive integer. The relationships between DT, solitonic specialization and the Hirota method have been presented in [15] for any hierarchy of integrable models possessing a zero curvature representation in terms of an affine KacMoody algebra. The DT method provides a relationship between the fields of the model and the relevant tau functions, and it explains the truncation of the Hirota expansion. The Hirota method is a recursive method

which can be implemented through a computer program for algebraic manipulation like MATHEMATICA. On the other hand, the DT method requires the computation of matrix elements as in eqs. (3.26)-(3.28) in the vertex operator representations of the affine KacMoody algebra. Actually, these matrix element calculations are very tedious in the case of higher soliton solutions.

3.1 Free field boundary conditions and dressing group

As a generalization of the constant NVBC (3.3) consider the free field NVBC

$$\lim_{x \rightarrow -\infty} \Psi_j^\pm \rightarrow \hat{\rho}_j^\pm(x, t) \equiv \rho_j^\pm e^{a_j^\pm x + b_j^\pm t} \quad (3.29)$$

where ρ_j^\pm , a_j^\pm , b_j^\pm are some constants. Notice that (3.29) is a free field solution of (2.6)-(2.7) such that $b_j^\pm = \pm(a_j^\pm)^2 \mp 2\Lambda_j$, for $\Lambda_j \equiv [\sum_{i=1}^2 \hat{\rho}_i^+ \hat{\rho}_i^- - \frac{1}{2}(\beta_j \cdot \vec{\Omega})] = \text{const}$. However, a direct application of the DT approach of [15] is not possible since the relevant connections \hat{A}^{vac} , \hat{B}^{vac} do not belong to an abelian subalgebra (up to the central term). So, in order to consider the NVBC (3.29) we resort to the dressing group [29] composition law of two successive DTs. In fact, the relevant connections can be related to the A^{vac} , B^{vac} in (3.6)-(3.7) through certain gauge transformations generated by Θ_\pm^g such that

$$\hat{V}_i^{vac} = \Theta_\pm^g(x, t) V_i^{vac} [\Theta_\pm^g(x, t)]^{-1} + \partial_{x_i} \Theta_\pm^g(x, t) [\Theta_\pm^g(x, t)]^{-1}; \quad \hat{V}_1 = \hat{A}^{vac}, \quad \hat{V}_2 = \hat{B}^{vac}, \quad (3.30)$$

where $i = 1, 2$, $x_1 = x$, $x_2 = t$, $V_1 = A^{vac}$, $V_2 = B^{vac}$, and $\Theta_\pm^g \in \widehat{SL}(3)$. One must have

$$\Psi^{(0)} \rightarrow \hat{\Psi}^{(0)} = \Theta_+^g(x, t) \Psi^{(0)} = \Theta_-^g(x, t) \Psi^{(0)} g, \quad \hat{V}_i^{(vac)} = \partial_{x_i} \hat{\Psi}^{(0)} [\hat{\Psi}^{(0)}]^{-1}, \quad (3.31)$$

g is an arbitrary constant group element, and

$$\Theta_+^g = e^{\chi_0} e^{\sum_{n>0} \chi_n}, \quad \Theta_-^g = e^{-\chi_0} e^{\sum_{n>0} \chi_{-n}}, \quad [D, \chi_n] = n \chi_n. \quad (3.32)$$

From (3.30) one has $e^{-\chi_0} E^{(l)} e^{\chi_0} = E^{(l)}$, so

$$\chi_0 = \chi_o^+ E_{\beta_1 - \beta_2}^{(0)} + \chi_o^- E_{-\beta_1 + \beta_2}^{(0)} + \sum_{a=1}^2 \chi_o^a H_a^{(0)} + \chi C \quad (3.33)$$

$$\chi_{-1} = -\sum_{i=1}^2 \left[f_i^+(x, t) - \rho_i^+ \right] E_{\beta_i}^{(-1)} + \sum_{i=1}^2 \left[f_i^-(x, t) - \rho_i^- \right] E_{-\beta_i}^{(-1)} + \sum_{a=1}^2 \chi_{-1}^a H_a^{(-1)}. \quad (3.34)$$

Therefore, a modified DT procedure can be implemented with [29]

$$[\Theta_-^{h'g}]^{-1} \Theta_+^{h'g} \equiv \hat{\Psi}^{(0)} h' [\hat{\Psi}^{(0)}]^{-1} = \Theta_-^g(x, t) \Psi^{(0)} h [\Psi^{(0)}]^{-1} [\Theta_+^g(x, t)]^{-1}; \quad h \equiv gh', \quad (3.35)$$

through

$$A' = \Theta_\pm^{h'g} A^{(vac)} [\Theta_\pm^{h'g}]^{-1} + \partial_x \Theta_\pm^{h'g} [\Theta_\pm^{h'g}]^{-1}; \quad B' = \Theta_\pm^{h'g} B^{(vac)} [\Theta_\pm^{h'g}]^{-1} + \partial_t \Theta_\pm^{h'g} [\Theta_\pm^{h'g}]^{-1}. \quad (3.36)$$

So, the equation (3.35) can be used instead of (3.14) in order to derive general dark-dark solitons with free field NVBC.

4 Bright solitons and vanishing boundary conditions

For simplicity we first apply the DT method to the VBC (3.2) and show the existence of bright-bright soliton solutions [30]. The connections $A^{vac} \equiv E^{(1)}$, $B^{vac} \equiv E^{(2)}$ are related to the group element Ψ_0 as

$$A^{(vac)} = \partial_x \Psi_0 [\Psi_0]^{-1}; \quad B^{(vac)} = \partial_t \Psi_0 [\Psi_0]^{-1}; \quad \Psi_0 \equiv e^{xE^{(1)} + tE^{(2)}}. \quad (4.1)$$

In order to obtain soliton solutions the simultaneous adjoint eigenstates of the elements $E^{(1)}$, $E^{(2)}$ play a central role. In the case at hand one has the eigenstates F_j and G_j in (C.1) such that

$$\left[xE^{(1)} + tE^{(2)}, F_j \right] = -\varphi_j(x, t)F_j; \quad \varphi_j(x, t) = \nu_j(x + \nu_j t) \quad (4.2)$$

$$\left[xE^{(1)} + tE^{(2)}, G_j \right] = \eta_j(x, t)G_j, \quad \eta_j(x, t) = \rho_j(x + \rho_j t) \quad (4.3)$$

4.1 j^{th} -component one bright-soliton solution

Consider the product

$$h = e^{a_{j_1} F_{j_1}} e^{b_{j_2} G_{j_2}}, \quad (4.4)$$

where j_1 and j_2 are some indexes chosen from $\{1, 2\}$. Using the nilpotency properties of F_{j_1} and G_{j_2} (see Appendix C) one gets

$$[\Psi_0 h \Psi_0^{-1}] = (1 + e^{-\varphi_{j_1}} a_{j_1} F_{j_1}) (1 + e^{\eta_{j_2}} b_{j_2} G_{j_2}) \quad (4.5)$$

$$= 1 + e^{-\varphi_{j_1}} a_{j_1} F_{j_1} + e^{\eta_{j_2}} b_{j_2} G_{j_2} + a_{j_1} b_{j_2} e^{-\varphi_{j_1}} e^{\eta_{j_2}} F_{j_1} G_{j_2}, \quad (4.6)$$

with φ_{j_1} and η_{j_2} given in (4.2)-(4.3). The corresponding tau functions become

$$\tau_0 = 1 + a_{j_1} b_{j_2} C_{j_1, j_2} e^{-\varphi_{j_1}} e^{\eta_{j_2}}, \quad C_{j_1, j_2} = \frac{\nu_{j_1} \rho_{j_2}}{(\nu_{j_1} - \rho_{j_2})^2} \delta_{j_1, j_2}, \quad (4.7)$$

$$\tau_i^+ = \delta_{i, j_2} b_{j_2} \rho_{j_2} e^{\eta_{j_2}}, \quad \tau_i^- = \delta_{i, j_1} a_{j_1} \nu_{j_1} e^{-\varphi_{j_1}}, \quad (4.8)$$

where the matrix element C_{j_1, j_2} has been presented in (C.2). In order to construct *one-soliton* solutions we must have $j_1 = j_2 \equiv j$ in (4.7). Therefore one gets ¹

$$\Psi_i^+ = \frac{b_i \rho_i e^{\eta_i}}{1 + a_i b_i C_{i, i} e^{-\varphi_i} e^{\eta_i}}, \quad \Psi_i^- = -\frac{a_i \nu_i e^{-\varphi_i}}{1 + a_i b_i C_{i, i} e^{-\varphi_i} e^{\eta_i}}, \quad i = j; \quad (4.9)$$

$$\Psi_i^\pm = 0, \quad i \neq j \quad (4.10)$$

Impose the relationships $\rho_j^* = -\nu_j$, $b_j^* = -\mu \delta a_j \nu_j$, $a_j \in \mathbb{C}$; so from (4.7) one has $C_{j, j} = -(\frac{\nu_j}{2\nu_{jR}})^2$. The equations (4.9)-(4.10) with the complexification (2.10) provide a solution of the CNLS system (2.11)

$$\psi_i(x, t) = \begin{cases} -\frac{(a_i \nu_i) \nu_{iR}}{\sqrt{\mu \delta |a_j \nu_j|^2}} e^{i\varphi_{iR}} \operatorname{sech} \left(\varphi_{iR} + \frac{X_0}{2} \right), & i = j \\ 0 & i \neq j \end{cases} \quad (4.11)$$

¹If $j_1 \neq j_2$ in (4.7)-(4.8) one still has certain trivial solutions, since in this case $C_{j_1, j_2} = 0$ implying $\tau_0 = 1$

where $e^{X_0} = \frac{\mu \delta |a_j \nu_j|^2}{(\nu_j + \nu_j^*)^2}$, $\varphi_j = \nu_j (x - i\nu_j t) \equiv \varphi_{jR} + i\varphi_{jI}$. It must be $\delta = +1$, and the solution (4.11) possesses 2 complex (a_j, ν_j) parameters plus the real coupling $\mu > 0$.

The solution (4.11) for the j 'th component is known as a 'bright soliton' in the context of the scalar non-linear Schrödinger equation (NLS).

4.2 1-bright-bright soliton solution

The main observation in the last construction of the j^{th} -component one-soliton is that it has been excited by the group element h in (4.4) such that $j_1 = j_2 = j$. So, in order to excite the two components of Ψ_i^\pm ($i = 1, 2$) and reproduce a bright soliton for each component let us consider the group element

$$h = e^{a_1 F_1} e^{b_1 G_1} e^{a_2 F_2} e^{b_2 G_2}, \quad (4.12)$$

where the exponential factors contain F 's and G 's of type (C.1). The tau functions become

$$\tau_j^+ = b_j \rho_j e^{\eta_j}, \quad \eta_j = \rho_j (x + \rho_j t) + \rho_{0j}, \quad j = 1, 2. \quad (4.13)$$

$$\tau_j^- = a_j \nu_j e^{-\varphi_j}, \quad \varphi_j = \nu_j (x + \nu_j t) + \nu_{0j}, \quad j = 1, 2. \quad (4.14)$$

$$\tau_0 = 1 + a_1 b_1 C_{11} e^{-\varphi_1} e^{\eta_1} + a_2 b_2 C_{22} e^{-\varphi_2} e^{\eta_2}, \quad (4.15)$$

where $C_{jj} = \frac{\rho_j \nu_j}{(\rho_j - \nu_j)^2}$. Let us set $b_j^* = -\mu \delta_j a_j$, $\rho_j^* = -\nu_j \equiv -\nu_1$, then $\eta_j^* = -\varphi_j \equiv -\varphi_1 = -(\varphi_{1R} + i\varphi_{1I})$, $(\nu_j, a_j \in \mathbb{C})$ in the relations (4.13)-(4.15). Therefore, the vector soliton solution of the 2-CNLS eq. (2.11) arises

$$\begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \nu_{1R} \operatorname{sech} \left(\varphi_{1R} + \frac{X_0}{2} \right) e^{i\varphi_{1I}}, \quad (4.16)$$

where $A_i = -\frac{a_i \nu_i}{\mu^{1/2} \sqrt{\sum_{j=1}^2 \delta_j |a_j \nu_j|^2}}$; $e^{X_0} = \mu \frac{\sum_j \delta_j |a_j \nu_j|^2}{(\nu_1 + \nu_1^*)^2}$.

This solution is valid for $\sum_j \delta_j |a_j \nu_j|^2 > 0$. Notice that this 1-bright-bright soliton possesses, apart from the real $\mu > 0$, four arbitrary complex parameters, namely, a_1, a_2, ν_1, ν_2 . This solution in the case of mixed nonlinearity $\delta_1 = -\delta_2 = 1$, may have a singular behavior if the sum $\sqrt{\sum_{j=1}^2 \delta_j |a_j \nu_j|^2}$ in the denominator of the expression of A_i vanishes, such that the soliton amplitude in (4.16) diverges. The N -bright-bright soliton requires the generalization of the group element in (4.12) as $h = e^{a_1 F_1} e^{b_1 G_1} \dots e^{a_N F_N} e^{b_N G_N}$.

5 AKNS_r ($r = 1, 2$), NVBC and dark solitons

In this section we tackle the problem of finding dark soliton type solutions of the system (2.6)-(2.7). The associated 2-CNLS model (2.11) with nonvanishing boundary conditions has been considered in the framework of direct methods, such as the Hirota tau function approach (see e.g. [10, 31, 22, 32]), and recently in the inverse scattering transform approach [12, 13]. In the last approach the relevant Lax operators have

remarkable differences and rather involved spectral properties as compared to their counterparts with vanishing boundary conditions (see [34] and references therein), e.g. in the NVBC case the spectral parameter requires the construction of certain Riemann sheets [34, 35, 36, 37]. So, it would be interesting to give the full Lie algebraic construction of the tau functions and soliton solutions for the system (2.6)-(2.7) with NVBC.

5.1 AKNS₁: N-dark solitons

For simplicity, firstly we describe the entire process for the system (2.6)-(2.7) with just two fields Ψ^\pm . So, let us consider the $\hat{sl}(2)$ affine Kac-Moody algebra in the Weyl-Cartan (WC) basis²

$$[H^{(m)}, H^{(n)}] = \frac{m}{2}\delta_{m+n,0}C, \quad [H^{(m)}, E_\pm^{(n)}] = \pm E_\pm^{(m+n)}, \quad [E_+^{(m)}, E_-^{(n)}] = 2H^{(m+n)} + m\delta_{m+n,0}C. \quad (5.1)$$

In order to study a NVBC for the system $\hat{sl}(2)$ -AKNS consider the Lax pair

$$A = E^{(1)} + \Psi^+ E_+^{(0)} + \Psi^- E_-^{(0)} + \phi_1 C, \quad (5.2)$$

$$B = E^{(2)} + \Psi^+ E_+^{(1)} + \Psi^- E_-^{(1)} + \partial_x \Psi^+ E_+^{(0)} - \partial_x \Psi^- E_-^{(0)} - 2(\Psi^+ \Psi^- - \rho^+ \rho^-) H^{(0)} + \phi_2 C, \quad (5.3)$$

where Ψ^+, Ψ^- are the fields of the model ($\rho^\pm = \text{constant}$). In this case one considers the generator $E^{(l)} \equiv H^{(l)}$. The ϕ_1 and ϕ_2 are introduced as auxiliary fields. Therefore the equations of motion suitable to treat NVBC become

$$\partial_t \Psi^+ = \partial_x^2 \Psi^+ - 2(\Psi^+ \Psi^- - \rho^+ \rho^-) \Psi^+, \quad (5.4)$$

$$\partial_t \Psi^- = -\partial_x^2 \Psi^- + 2(\Psi^+ \Psi^- - \rho^+ \rho^-) \Psi^-, \quad (5.5)$$

$$\partial_t \phi_1 - \partial_x \phi_2 = 0. \quad (5.6)$$

The AKNS₁ model (5.4)-(5.5) is recovered from the AKNS₂ extension (2.6)-(2.7) simply by setting to zero the additional fields. In [33] it has been introduced a complexified version of the system (5.4)-(5.5) (for $\rho^\pm = 0$) and presented its reduction to the focusing and defocusing NLS system, as well as its soliton solutions. In fact, the $\hat{sl}(2)$ -AKNS model (5.4)-(5.5) as well, through a particular reduction, contains as sub-model the scalar defocusing NLS system

$$i \partial_t \psi + \partial_x^2 \psi - 2(|\psi|^2 - \rho^2) \psi = 0. \quad (5.7)$$

This equation is suitable for treating nonvanishing boundary conditions (NVBC) [37, 34, 35]

$$\psi(x, t) = \begin{cases} \rho, & x \rightarrow -\infty \\ \rho \epsilon^2, & x \rightarrow +\infty \end{cases}; \quad \epsilon = e^{i\theta/2}, \quad \rho \in \mathbb{R}. \quad (5.8)$$

In fact, this boundary condition is manifestly compatible with the eq. of motion (5.7). The defocusing NLS equation (5.7) with the boundary condition (5.8) is exactly integrable by the inverse scattering technique

²This basis differs from the Chevalley (Ch) basis in (A.1)-(A.4) by the rescaling of the generator $H_{WC}^{(m)} \rightarrow \frac{1}{2}H_{Ch}^{(m)}$.

[38]. This model has soliton solutions in the form of localized “dark” pulses created on a constant or stable *continuous* wave background solution.

In the system of eqs. (5.4)-(5.5) consider the transformation

$$t \rightarrow -it, \quad x \rightarrow -ix, \quad (5.9)$$

$$\Psi^\pm \rightarrow i \Psi^\pm \epsilon^{\mp 2}, \quad (5.10)$$

where the factor $\epsilon^{\mp 2}$ is introduced for later convenience. Furthermore, the identification

$$\psi \equiv \Psi^+ = (\Psi^-)^*, \quad (5.11)$$

where the star stands for complex conjugation, such that $\rho^+ \rho^- \rightarrow -\rho^2$ allows one to reproduce the defocusing NLS equation (5.7).

Let us take as vacuum solution of (5.4)-(5.5) the constant background configuration

$$\Psi^\pm = \rho^\pm \equiv \rho \epsilon^{\mp 2}, \quad \phi_{1,2} = 0, \quad \rho, \epsilon = \text{const.} \quad (5.12)$$

Therefore the gauge connections (5.2)-(5.3) for the vacuum solution (5.12) become

$$A^{vac} \equiv \varepsilon_1 = E^{(1)} + \rho^+ E_+^{(0)} + \rho^- E_-^{(0)} \quad (5.13)$$

$$B^{vac} = \varepsilon_2 = E^{(2)} + \rho^+ E_+^{(1)} + \rho^- E_-^{(1)} \quad (5.14)$$

We follow eq. (3.8) in order to write the connections in the form

$$A^{(vac)} = \partial_x \Psi_{nvbc}^{(0)} [\Psi_{nvbc}^{(0)}]^{-1}, \quad B^{(vac)} = \partial_t \Psi_{nvbc}^{(0)} [\Psi_{nvbc}^{(0)}]^{-1}, \quad (5.15)$$

with the group element

$$\begin{aligned} \Psi_{nvbc}^{(0)} &\equiv e^{x\varepsilon_1 + t\varepsilon_2} \\ &= \exp \left[x(E^{(1)} + \rho^+ E_+^{(0)} + \rho^- E_-^{(0)}) + t(E^{(2)} + \rho^+ E_+^{(1)} + \rho^- E_-^{(1)}) \right] \end{aligned} \quad (5.16)$$

Let us emphasize that this group element differs fundamentally from the one associated to the case with VBC. First, the difference originates from the constant boundary condition terms added to the relevant vacuum gauge connections. However, they must be related since $\lim_{\rho \rightarrow 0} \Psi_{nvbc}^{(0)} \rightarrow \Psi_0$, where Ψ_0 is the group element in the VBC case (4.1). Second, the vacuum connections $\varepsilon_{1,2}$ in the DT procedure will require another eigenvectors under the adjoint actions in analogy to eqs. (4.2)-(4.3), as we will see below.

Another proposal for this group element $\Psi_{nvbc}^{(0)}$ has been introduced in [39]. There it has been considered a group element inspired in the inverse scattering method, possessing double-valued functions of the spectral parameter λ . This fact motivates the introduction of an affine parameter to avoid constructing Riemann sheets. This construction involves some complications when used in the context of the DT method. The main difficulty arises in the computation of the matrix elements associated to the highest weight states in

order to find the relevant tau functions. However, the group element given in (5.16) will turn out to be more suitable in the DT procedure. In fact, a similar group element has been proposed in [40] to tackle a NVBC soliton solutions in the so-called negative even grade mKdV hierarchy.

The DT procedure in this case follows all the way verbatim as in section 3 adapted to the $sl(2)$ case. It follows by relating the relevant connections (5.2)-(5.3) with the connections in eqs. (5.13)-(5.14) corresponding to the vacuum solution (5.12).

Next, we will write the fields Ψ^\pm in terms of the relevant tau functions. According to the development in section 3 these functions will be written as certain matrix elements in an integrable highest weight representation of the affine Lie algebra $\widehat{sl}(2)$. So, it is a straightforward task to get the following relationships

$$\Psi^+ = \rho^+ + \frac{\tau^+}{\tau_0} \quad \text{and} \quad \Psi^- = \rho^- - \frac{\tau^-}{\tau_0} \quad , \quad (5.17)$$

where the *tau* functions τ^\pm , τ_0 are given by

$$\tau^+ \equiv \langle \lambda_o | E_-^{(1)} \left[\Psi_{nvbc}^{(0)} h \Psi_{nvbc}^{(0)-1} \right]_{(-1)} | \lambda_o \rangle , \quad (5.18)$$

$$\tau^- \equiv \langle \lambda_o | E_+^{(1)} \left[\Psi_{nvbc}^{(0)} h \Psi_{nvbc}^{(0)-1} \right]_{(-1)} | \lambda_o \rangle , \quad (5.19)$$

$$\tau_0 \equiv \langle \lambda_o | \left[\Psi_{nvbc}^{(0)} h \Psi_{nvbc}^{(0)-1} \right]_{(o)} | \lambda_o \rangle , \quad (5.20)$$

with the group element $\Psi_{nvbc}^{(0)}$ given by (5.16).

5.1.1 AKNS₁ and 1-dark soliton of defocusing scalar NLS

Let us consider the group element

$$h^q = e^{a^q V^q(\gamma, \rho^\pm)}, \quad (q = 1, 2) \quad (5.21)$$

where a^q is a constant. The vertex operator V^q satisfies $\hat{V}^q = 2V^q$ (see below) where \hat{V}^q is defined in (A.10).

Next, one must look for the eigenvalues of ε_1 and ε_2 , respectively, in their adjoint actions on the vertex operator V^q . In order to achieve this, let us notice that these Weyl-Cartan basis elements $\varepsilon_{1,2}$ are related to the corresponding Chevalley basis elements $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ in (A.12) through $\hat{\varepsilon}_n = 2\varepsilon_n$, ($n = 1, 2$), provided that $\hat{\rho}^\pm = 2\rho^\pm$ and the relationship between the Weyl-Cartan and Chevalley basis $H_{WC}^{(n)} = \frac{1}{2}H_{Ch}^{(n)}$ is considered for the $sl(2)$ element $H^{(n)}$. So, the vertex operators are related by $\hat{V}^q = 2V^q$. Then from (A.11) one has

$$[\varepsilon_1, V^q] = \gamma V^q, \quad [\varepsilon_2, V^q] = (-1)^{q-1} \gamma (\gamma^2 - 4\rho^+ \rho^-)^{1/2} V^q, \quad q = 1, 2. \quad (5.22)$$

Notice that these eigenstates V^q exhibit a more complex structure in comparison to their counterparts F_j and G_j corresponding to the VBC as in (4.2)-(4.3). Therefore one has

$$[x\varepsilon_1 + t\varepsilon_2, V^q(\gamma, \rho^\pm)] = [\gamma(x + (-1)^{q-1}vt)] V^q(\gamma_i, \rho^\pm), \quad (5.23)$$

where $v = \sqrt{\gamma^2 - 4\rho^+ \rho^-}$. Denoting $\varphi_q = \gamma(x + (-1)^{q-1}vt)$ one can write

$$\left[\Psi_{nvbc}^{(0)} h \Psi_{nvbc}^{(0)-1} \right] = \exp(e^{\varphi_q} a_q V^q) \quad (5.24)$$

$$= 1 + e^{\varphi_q} a_q V^q, \quad (5.25)$$

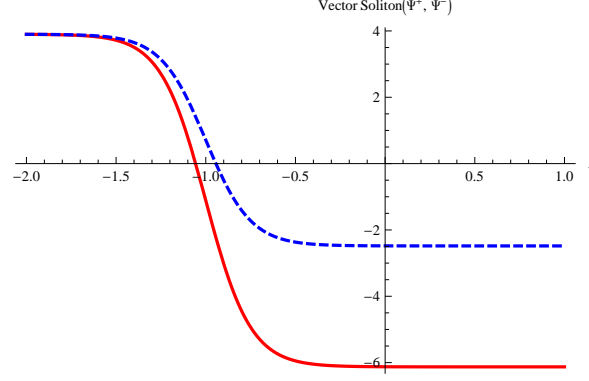


Figure 1: The real vector (Ψ^+, Ψ^-) 1-soliton profile of the AKNS₁ model for $q = 1$ in (5.26). They were plotted for $\gamma_1 = 8$, $\rho^+ = \rho^- = 3.9$, $\log c^{(1)} = 8$. The solid and dashed lines correspond to Ψ^+ and Ψ^- , respectively.

where we have used $(V^q)^n = 0$, for $n \geq 2$, that is, V^q is nilpotent (A.17)-(A.18).

So, in order to find the explicit tau functions in (5.18)-(5.20) it remains to compute the relevant matrix elements. Using the properties presented in the Appendix A and the eqs. (3.26)-(3.28) one gets the tau functions $\tau_{(q)}^0 = 1 + c^{(q)} e^{\varphi_q}$; $\tau_{(q)}^{\pm} = s^{\pm(q)} \rho^{\pm} e^{\varphi_q}$, where the matrix elements $c^{(q)} = a^{(q)} \langle \lambda_o | V^q | \lambda_o \rangle$, $s^{\pm(q)} = a^{(q)} (\rho^{\pm})^{-1} \langle \lambda_o | E_{\mp}^{(1)} V^q | \lambda_o \rangle$ have been considered. They are given in the Appendix A, eqs. (A.13)-(A.14), respectively. So, one has $\frac{s^{\pm(q)}}{2c^{(q)}} = \frac{\gamma}{4\rho^+\rho^-} \left(\gamma \pm (-1)^{q-1} \sqrt{\gamma^2 - 4\rho^+\rho^-} \right)$. Without loss of generality we can assume $\gamma > 0$, so, the matrix element $\langle \lambda_o | V^q | \lambda_o \rangle$ from (A.13) allows us to determine the sign of $c^{(q)}$, which is completely fixed by $\text{sign}[c^{(q)}] = e_q \text{sign}[a^{(q)}]$. We will see below that the signs of the free real parameter $c^{(q)}$ determine two different type of solutions.

Consider $\mathbf{c}^{(q)} > \mathbf{0}$. This case follows if $a^{(q)}$ and e_q have equal sign. Then, the relations (5.17) supplied with the above tau functions provide

$$\Psi^{\pm(q)}(x, t) = \rho^{\pm} \left[1 - \frac{s^{\pm(q)}}{2c^{(q)}} \right] - \frac{\rho^{\pm} s^{\pm(q)}}{2c^{(q)}} \tanh \left\{ \gamma [x + (-1)^{q-1} vt] / 2 + \frac{\log c^{(q)}}{2} \right\}. \quad (5.26)$$

These solutions are the vector 1-soliton of the AKNS₁ model. This solution possesses 4 real parameters, i.e. ρ^{\pm} , γ , $c^{(q)}$. Notice that the value of the index q determines the direction of propagation of the soliton. The Fig. 1 displays the component solitons of this vector 1-soliton. The true 1-dark soliton of the AKNS₁ model is obtained as the product

$$[\Psi^{-(q)} \Psi^{+(q)}](x, t) = \rho^+ \rho^- - \frac{\gamma^2}{4} \text{sech}^2 \left\{ \gamma [x + (-1)^{q-1} vt] / 2 + \frac{\log c^{(q)}}{2} \right\}. \quad (5.27)$$

It has been plotted in Fig. 2 for the case $q = 1$.

Next, consider $\mathbf{c}^{(q)} < \mathbf{0}$. This case follows if $a^{(q)}$ and e_q have opposite sign. Similarly, the relations (5.17) and the above tau functions provide

$$\Psi^{\pm(q)}(x, t) = \rho^{\pm} \left[1 - \frac{s^{\pm(q)}}{2c^{(q)}} \right] - \frac{\rho^{\pm} s^{\pm(q)}}{2c^{(q)}} \coth \left\{ \gamma [x + (-1)^{q-1} vt] / 2 + \frac{\log |c^{(q)}|}{2} \right\}. \quad (5.28)$$

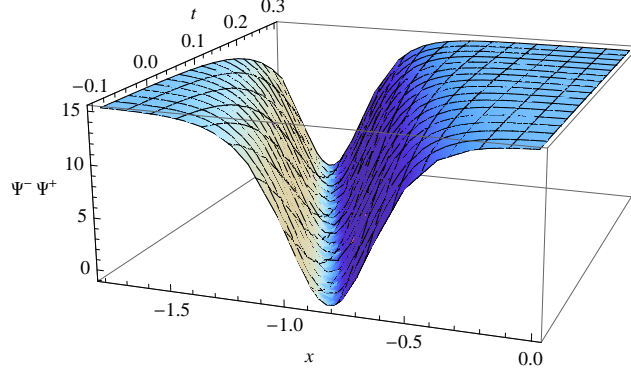


Figure 2: The 1-dark soliton profile of the AKNS₁ model emerges as $\Psi^-\Psi^+$. This is the soliton traveling to the left, the case $q = 1$ in (5.27). It was plotted for $\gamma_1 = 8$, $\rho^+ = \rho^- = 3.9$, $\log c^{(1)} = 8$.

These solutions are the singular solitons of the AKNS₁ model and, to our knowledge, they have never been reported yet. As above the value of the index q determines the direction of propagation of the soliton. They possess 4 real parameters, namely, ρ^\pm , γ , $c^{(q)}$. Similarly, let us consider the product of the components

$$[\Psi^{-(q)}\Psi^{+(q)}](x,t) = \rho^+\rho^- - (-1)^{q-1}\frac{\gamma^2}{4}\text{csch}^2\left\{\gamma[x + (-1)^{q-1}vt]/2 + \frac{\log|c^{(1)}|}{2}\right\} \quad (5.29)$$

These functions are displayed in Fig. 3. Notice that these solitons are singular for $[\gamma[x + (-1)^{q-1}vt]/2 + \frac{\log|c^{(1)}|}{2}] = 0$.

Some comments are in order here.

First, the 1-dark soliton solutions (5.27) and the singular solitons (5.29), respectively, have the same boundary values in $x \rightarrow \pm\infty$, i.e. $\lim_{|x| \rightarrow +\infty} \Psi^{-(q)}(x,t)\Psi^{+(q)}(x,t) \rightarrow \rho^+\rho^-$. In the both types of solitons this behavior is in contrast with the different boundary values assumed by the corresponding components in the both limits. For the 1-dark soliton components one has $\lim_{x \rightarrow -\infty} \Psi^\pm \rightarrow \rho^\pm$, whereas $\lim_{x \rightarrow +\infty} \Psi^\pm \rightarrow \rho^\pm[1 - \frac{\gamma}{2\rho^+\rho^-}(\gamma \pm \sqrt{\gamma^2 - 4\rho^+\rho^-})]$. This fact is shown in Fig. 1.

Second, the center of the 1-dark solitons are given for $\{\gamma[x + (-1)^{q-1}vt]/2 + \frac{\log c^{(q)}}{2}\} \equiv 0$. In fact, in this point the functions $\Psi^{-(q)}\Psi^{+(q)}|_{center} = \rho^+\rho^- - \gamma^2/4$ (for $q = 1, 2$) take the smallest intensity. So, their intensity dips are controlled by the value of the parameter γ .

Third, the solution (5.26) in the case $q = 1$ allows the imposition of the conditions (5.11) in order to satisfy the defocusing NLS equation (5.7). So, let us make the transformation (5.9) and the complexification $\gamma_1 \rightarrow i\gamma$, $\rho \rightarrow i\rho$, $v_\gamma \rightarrow \bar{v}_\gamma = \sqrt{4\rho^2 - \gamma^2}$; ($2\rho > \gamma$). Choose $a_1 = -i\gamma/\bar{v}$. Moreover, considering the transformation (5.10) one gets the functions satisfying the complexification condition (5.11). Therefore, the function $\psi(x,t) \equiv -i\epsilon^2\Psi^+$, where $\Psi^+ = i\epsilon^{-2}\{\rho + (\frac{i\gamma\rho}{\bar{v}_\gamma - i\gamma})\frac{e^{\bar{\varphi}(x,t)}}{1 + e^{\bar{\varphi}(x,t)}}\}$, $\bar{\varphi}(x,t) = \gamma(x - \bar{v}_\gamma t) + \gamma_0$, obtained in this way is a solution of the defocusing NLS equation (5.7). In fact, the function ψ can be written as

$$\psi(x,t) = e^{i(\alpha + \pi/2)}\left\{\frac{\gamma}{2}\tanh[\gamma(x - \bar{v}_\gamma t)/2 + \gamma_0/2] + \frac{\gamma}{2} - i\rho e^{-i\alpha}\right\}, \quad \alpha = \tan^{-1}\left(\frac{\gamma}{\bar{v}_\gamma}\right), \quad (5.30)$$

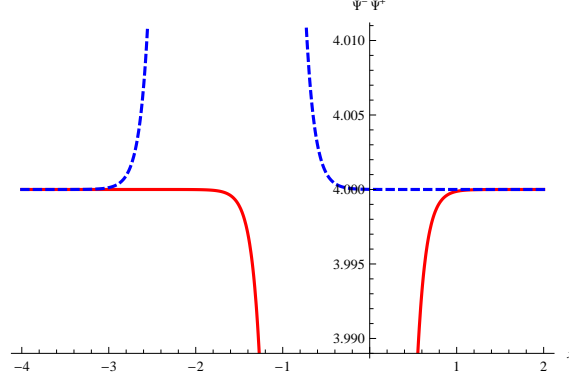


Figure 3: The singular solitons of the AKNS₁ model. The solid lines travel to the left ($q = 1$) and the dashed lines to the right ($q = 2$). They were plotted for $t = -0.07$, $\gamma = 10$, $\rho^+ = \rho^- = 2$, $\log|c^{(q)}| = 10$.

which is the dark soliton solution of the defocusing NLS equation (5.7) (see e.g. [31]). Notice that this solution satisfy the NVBC $\lim_{x \rightarrow \pm\infty} |\psi(x, t)|^2 \rightarrow \rho^2$.

Fourth, similarly, one can consider the complexification condition (5.11) in the case $q = 2$. In this case one gets $\Psi^\pm = i\epsilon^{\mp 2}\rho \left[\mp \frac{\gamma}{2\rho} e^{\pm i\delta} \tanh[(\gamma x + \gamma \bar{v}_\gamma t)/2] + 1 \mp \frac{\gamma}{2\rho} e^{\pm i\delta} \right]$, $\delta = \tan^{-1}(\bar{v}_\gamma/\gamma)$. However, they do not satisfy the condition (5.11), hence these functions would not be solutions of (5.7).

From the previous constructions it is clear that the N-solitons of the types dark, singular or mixed dark-singular can be constructed by choosing convenient signs of the parameters $a_j^{(q)}$ and e_q associated to the group element

$$h = e^{a_1^{q_1} V^q(\gamma^1, \rho^\pm)} e^{a_2^{q_2} V^q(\gamma^2, \rho^\pm)} \dots e^{a_N^{q_N} V^q(\gamma^N, \rho^\pm)}, \quad q_j = 1, 2 \quad (5.31)$$

5.2 AKNS₂: N-dark-dark solitons

Consider the non-vanishing boundary condition (NVBC) for the system (2.6)-(2.7) in the form (3.1) and its associated constant vacuum solution (3.3). The vacuum connections (3.6)-(3.7) associated to this trivial solution commute $[A^{vac}, B^{vac}] = 0$ if one takes $\Omega_1 = \Omega_2 = \frac{2}{3}(\rho_1^+ \rho_1^- + \rho_2^+ \rho_2^-)$. Consider the notation $A^{vac} = \varepsilon_1$, $B^{vac} = \varepsilon_2$.

1-dark-dark soliton. Let us consider the eqs. (3.26)-(3.28) and choose the group element $h^q = e^{a^q W^q(k)}$, a^q and k^q being some parameters, where the vertex operator $W^q(k)$ is defined in (C.7), such that

$$[\varepsilon_1, W^q] = k^q W^q; \quad q = 1, 2; \quad (5.32)$$

$$[\varepsilon_2, W^q] = w^q W^q \quad (5.33)$$

where

$$w^q = e_q k^q \sqrt{(k^q)^2 - 4\rho_1^+ \rho_1^- - 4\rho_2^+ \rho_2^-}, \quad e_q \equiv (-1)^{q-1} \quad (5.34)$$

So, one has

$$[x\varepsilon_1 + t\varepsilon_2, W^q] = (k^q x + w^q t)W^q \quad (5.35)$$

The vertex operator W^q (see (C.7)) is associated to the 1-dark-dark soliton and it can be shown to be nilpotent (C.8). So, using (3.26)-(3.28) the following tau functions correspond to the element h^q given above

$$\tau_0^{(q)} = 1 + c^{(q)} e^{k^q x + w^q t}, \quad (5.36)$$

$$\tau_i^{\pm(q)} = \rho_i^{\pm} s^{\pm(q)} e^{k^q x + w^q t}, \quad i = 1, 2; \quad (5.37)$$

where $c^{(q)} = a^q < \lambda_o |W^q| \lambda_o >$, $s^{\pm(q)} = (\rho_i^{\pm})^{-1} a^q < \lambda_o |E_{\mp\beta_i}^{(1)} W^q| \lambda_o >$. The above tau functions provide a 1-dark-dark solution for the fields Ψ_i^{\pm} of the system (2.6)-(2.7), such that the following relationships hold between the parameters

$$c^{(q)} = \frac{s^{+(q)} s^{-(q)}}{s^{+(q)} - s^{-(q)}}; \quad (k^q)^2 = -\frac{(s^{+(q)} - s^{-(q)})^2 (\sum_k \rho_k^+ \rho_k^-)}{s^{+(q)} s^{-(q)}}; \quad w^q = \frac{s^{+(q)} + s^{-(q)}}{s^{+(q)} - s^{-(q)}} (k^q)^2. \quad (5.38)$$

We will concentrate on the regular solutions of the AKNS₂ model since they will give rise to the dark solitons, as it was clear from the AKNS₁ construction. So, in what follows we will require $c^{(q)} > 0$. Then, the relations (3.25) provide

$$\Psi_j^{\pm(q)} = \frac{\rho_j^{\pm}}{2} \left[(1 + (y^q)^{\pm 1}) + (-1 + (y^q)^{\pm 1}) \tanh\{(k^q x + w^q t + \log c^{(q)})/2\} \right], \quad (5.39)$$

where $j = 1, 2$; $y^q \equiv \frac{s^{+(q)}}{s^{-(q)}}$. The condition $c^{(q)} > 0$ requires that $y^q > 1$ and $y^q < 0$. This vector soliton possesses 6 real parameters, i.e. ρ_j^{\pm} , k^q , $c^{(q)}$.

Taking into account (C.7) one can compute the relevant matrix elements, which define $s^{\pm(q)}$, in order to get

$$y^q = -\frac{k^q + e_q \sqrt{(k^q)^2 - 4 \sum_j \rho_j^+ \rho_j^-}}{k^q - e_q \sqrt{(k^q)^2 - 4 \sum_j \rho_j^+ \rho_j^-}}. \quad (5.40)$$

Notice that for any value of the index q , if $\sum_j \rho_j^+ \rho_j^- > 0$ the last relationship implies $y^q < 0$ and if $\sum_j \rho_j^+ \rho_j^- < 0$ then $y^q > 0$.

Some remarks are in order here.

First, the solitons associated to the pair of components (Ψ_1^+, Ψ_2^+) and (Ψ_1^-, Ψ_2^-) , respectively, are proportional, so they are degenerate and reducible to the single dark soliton of the $sl(2)$ AKNS₁ model (5.26). Whereas, the solitons associated to the pair of components (Ψ_i^+, Ψ_j^-) , respectively, are not proportional, so they are non-degenerate presenting in general different degrees of ‘darkness’ in each component. In this way the solution in (5.39) presents a non-degenerate 1-dark-dark soliton in the components, say (Ψ_1^+, Ψ_2^-) of the AKNS₂ model. A nondegenerate 1-dark-dark soliton in the defocusing 2-CNLS has been given in [22, 10] through the Hirota’s method. The inverse scattering [12, 13] and Hirota’s method [10] results on N-dark-dark solitons in the defocusing 2-CNLS model have presented only the degenerate case.

Second, as in the AKNS₁ solution (5.26), one can show that the 1-dark-dark solitons (5.39) allow the imposition of the conditions (2.10) for the pair of fields (Ψ_i^+, Ψ_i^-) , $i = 1, 2$, in order to satisfy the 2-CNLS equation (2.11). In fact, consider the parameters $y^q = \frac{z^q+1}{z^q-1}$ and a complexification procedure as in (2.10) provided with a convenient set of complex parameters in (5.39), $k^q \rightarrow ik^q$, $\rho_i^\pm \rightarrow i\rho_i^\pm$ such that $y^q = e^{2i\phi_q}$, $z^q = \frac{e_q}{ik^q} \sqrt{4 \sum_j \rho_j^+ \rho_j^- - (k^q)^2}$, ϕ_q being real, so the equation (2.10) can be satisfied if

$$(\rho_j^+)^* = \mu \delta_j \rho_j^-. \quad (5.41)$$

Notice that, as it was shown in the AKNS₁ case and its 1-dark soliton (5.30) particular reduction, in order to get 2-CNLS 1-dark solitons one must consider $q = 1$ in the both components (Ψ_j^+, Ψ_j^-) . Therefore, the relationship between the parameters ρ_i^\pm determine clearly if these solutions will correspond to the defocusing ($\delta_j = -1$) or to the mixed nonlinearity ($\delta_1 = -\delta_2 = \pm 1$) 2-CNLS system, respectively.

Third, let us discuss the extension of our results to the case of free field NVBC (3.29). In order to compute the relevant tau functions analog to the ones in (3.26)-(3.28) one must consider the expression (3.35) instead of (3.14). So, the tau functions become

$$\hat{\tau}_i^{+g}(x, t) \equiv \langle \lambda_o | E_{-\beta_i}^{(1)} \Theta_-^g(x, t) \Psi^{(0)} h[\Psi^{(0)}]^{-1} | \lambda_o \rangle e^{-\chi}, \quad (5.42)$$

$$= (\hat{\rho}_i^+(x, t) - \rho_i^+) \hat{\tau}_0^g(x, t) + \langle \lambda_o | \left[e^{-\chi_o} E_{-\beta_i}^{(1)} e^{\chi_o} \right] \Psi^{(0)} h[\Psi^{(0)}]^{-1} | \lambda_o \rangle \quad (5.43)$$

$$\hat{\tau}_i^{-g}(x, t) \equiv \langle \lambda_o | E_{\beta_i}^{(1)} \Theta_-^g(x, t) \Psi^{(0)} h[\Psi^{(0)}]^{-1} | \lambda_o \rangle e^{-\chi}, \quad (5.44)$$

$$= (-\hat{\rho}_i^-(x, t) + \rho_i^-) \hat{\tau}_0^g(x, t) + \langle \lambda_o | \left[e^{-\chi_o} E_{\beta_i}^{(1)} e^{\chi_o} \right] \Psi^{(0)} h[\Psi^{(0)}]^{-1} | \lambda_o \rangle \quad (5.45)$$

$$\hat{\tau}_0^g(x, t) \equiv \langle \lambda_o | \Psi^{(0)} h[\Psi^{(0)}]^{-1} | \lambda_o \rangle. \quad (5.46)$$

where the group element $\Theta_-^g(x, t)$ is defined in (3.31)-(3.32) and $\chi(x, t)$ is an ordinary function. We have used Θ_+^g from eq. (3.32) and the decomposition (3.33) for χ_o , as well as the properties (B.15)-(B.16) of the highest weight representation. The tau functions (5.42)-(5.46) exhibit the modifications to be done on the previous eqs. (3.26)-(3.28) in order to satisfy the NVBC (3.29). These amount to introduce the group element Θ_-^g and the factor $e^{-\chi}$. Then from (5.42)-(5.46) and the analog to (3.25) $\Psi_i^+ = \rho_i^+ + \frac{\hat{\tau}_i^{+g}}{\hat{\tau}_0^g}$ and $\Psi_i^- = \rho_i^- - \frac{\hat{\tau}_i^{-g}}{\hat{\tau}_0^g}$ one can get

$$\Psi_i^+ = \hat{\rho}_i^+(x, t) + \frac{\hat{\tau}_i^+}{\hat{\tau}_0}; \quad \Psi_i^- = \hat{\rho}_i^-(x, t) - \frac{\hat{\tau}_i^-}{\hat{\tau}_0} \quad (5.47)$$

$$\hat{\tau}_i^\pm \equiv \langle \lambda_o | \left[e^{-\chi_o} E_{-\beta_i}^{(1)} e^{\chi_o} \right] \Psi^{(0)} h[\Psi^{(0)}]^{-1} | \lambda_o \rangle; \quad \hat{\tau}_0^g \equiv \hat{\tau}_0 = \langle \lambda_o | \Psi^{(0)} h[\Psi^{(0)}]^{-1} | \lambda_o \rangle \quad (5.48)$$

In fact, the relations (5.47) can be formally obtained from the ones in (3.25) by making the changes $\rho_i^\pm \rightarrow \hat{\rho}_i^\pm(x, t)$, $\tau_i^\pm \rightarrow \hat{\tau}_i^\pm$, $\tau_0 \rightarrow \hat{\tau}_0$. The soliton type solutions can be obtained following similar steps as the previous ones; i.e. for 1-soliton choose $h^q = e^{a^q W^q(k)}$ and the adjoint eigenvectors of $\varepsilon_{1,2}$ become precisely the vertex operator W^q as in (5.32)-(5.33), then the expressions $(k^q x + w^q t)$ of the tau functions (5.36)-(5.37) do not change.

Next, let us describe the general properties of the solutions (5.47)-(5.48). i) Notice that the form of $\hat{\tau}_0$ remains the same as the one in (5.36). ii) The modified DT and the related tau functions $\hat{\tau}_\pm$ (5.48) will

introduce some new parameters into the tau functions τ_i^\pm in (5.37). In (5.37) the factor ρ_i^\pm must be replaced by $\hat{\rho}_i^\pm(x, t)$ and $s^{\pm(q)}$ in general will depend on the index 'i' and the space-time, i.e. $s^{\pm(q)} \rightarrow s^{\pm(q, i)}(x, t)$. The explicit form one gets from $s^{\pm(q, i)}(x, t) = (\hat{\rho}_i^\pm(x, t))^{-1} a^q < \lambda_o | [e^{-\chi_o} E_{\mp \beta_i}^{(1)} e^{\chi_o}] W^q | \lambda_o >$. In the case of 1-dark-dark-soliton it is required an element χ_o such that $s^{\pm(q, i)}(x, t) \equiv s^{\pm(q)} \chi_i^{\pm(q)}$, $s^{\pm(q)}$, $\chi_i^{\pm(q)}$ being constant parameters. So, under the above modified DT one has the general form of the 1-dark-dark soliton

$$\Psi_j^{\pm(q)} = \frac{\rho_j^\pm e^{a_j^\pm x + b_j^\pm t}}{2} \left[(1 + (y_j^q)^{\pm 1}) + (-1 + (y_j^q)^{\pm 1}) \tanh\{(k^q x + w^q t + \log c_j^{(q)})/2\} \right], \quad (5.49)$$

where $j = 1, 2$; $y_j^q \equiv \frac{s^{+(q, j)}}{s^{-(q, j)}}$, and

$$b_i^\pm = \pm(a_i^\pm)^2 \mp 2 \left[\sum_{j=1}^2 \rho_j^+ \rho_j^- - \frac{1}{2}(\beta_i \cdot \vec{\Omega}) \right]; \quad \beta_1 \cdot \vec{\Omega} \neq \beta_2 \cdot \vec{\Omega}. \quad (5.50)$$

The solutions (5.49) are the general 1-dark-dark soliton components of the AKNS₂ model. The index $q = 1, 2$ refers to the solitons traveling to the left and right, respectively, and this solution has 13 real parameters, namely, ρ_j^\pm , k^q , $c_j^{(q)}$, a_j^\pm , Ω_1 , Ω_2 ; i.e. additional 7 real parameters as compared with the solution in (5.39). A complexified version of this soliton for $q = 1$, which is a solution of the 2-CNLS model, has recently been reported in [4]. Moreover, the general form has also been reported in [14] for AKNS_r using another approach. As in the AKNS₁ case (5.27), it is clear that the 1-dark soliton profiles can be recovered by plotting the functions $[\Psi_j^{-(q)} \Psi_j^{+(q)}](x, t)$, ($j = 1, 2$; $q = 1, 2$).

Moreover, similar steps can be followed in order to get the singular solutions of the AKNS₂ model related to $c_j^{(q)} < 0$. They will give rise to singular solitons as in the case of the AKNS₁ construction (5.28)-(5.29). It is also possible to construct mixed dark-singular solitons by conveniently chosen the positive and negative values of the parameters $c_j^{(q)}$ of each component of the vector (Ψ_j^\pm) .

2-dark-dark solitons. In order to get the two-dark-dark soliton ($N = 2$) solution one must choose the group element $h_2^q = e^{a_1 W^q(k_1)} e^{a_2 W^q(k_2)}$. Then following similar steps one has

$$\tau_0 = 1 + c_1 e^{k_1 x + w_1 t} + c_2 e^{k_2 x + w_2 t} + c_1 c_2 d_0 e^{k_1 x + w_1 t} e^{k_2 x + w_2 t}, \quad (5.51)$$

$$\tau_i^\pm = \rho_i^\pm \left[s_1^\pm e^{k_1 x + w_1 t} + s_2^\pm e^{k_2 x + w_2 t} + s_1^\pm s_2^\pm d^\pm e^{k_1 x + w_1 t} e^{k_2 x + w_2 t} \right], \quad i = 1, 2 \quad (5.52)$$

where

$$d_0 = \left(\frac{\sqrt{s_1^+ s_2^-} - \sqrt{s_1^- s_2^+}}{\sqrt{s_1^- s_2^-} - \sqrt{s_1^+ s_2^+}} \right)^2; \quad d^\pm = -\frac{(s_1^- s_2^- - s_1^+ s_2^+)}{(s_1^- - s_1^+)(s_2^- - s_2^+)} d_0; \quad (5.53)$$

$$c_n = \frac{s_n^+ s_n^-}{s_n^+ - s_n^-}; \quad (k_n)^2 = -\frac{(s_n^+ - s_n^-)^2 (\sum_{k=1}^2 \rho_k^+ \rho_k^-)}{s_n^+ s_n^-}; \quad w_n = \frac{s_n^+ + s_n^-}{s_n^+ - s_n^-} k_n^2, \quad (n = 1, 2) \quad (5.54)$$

Notice that the above tau functions related to two-dark-dark soliton must require $c_j > 0$. This two-soliton has 8 real parameters, i.e. ρ_j^\pm , k_1 , k_2 , c_1 , c_2 . The above process can be extended for N -dark-dark soliton solutions, in that case the group element in (3.26)-(3.28) must take the form $h_2^q = e^{a_1 W_1^q(k_1)} e^{a_2 W_2^q(k_2)} \dots e^{a_N W_N^q(k_N)}$.

The relevant tau functions follow

$$\tau_0 = \left| \delta_{mn} + \frac{s_m^+ s_n^-}{\sqrt{s_m^+ s_n^+} - \sqrt{s_m^- s_n^-}} e^{k_n x + w_n t} \right|_{N \times N}; \quad m, n = 1, 2, 3 \dots N \quad (5.55)$$

$$\tau_i^\pm = \rho_i^\pm \left\{ \left| \delta_{mn} + s_n^\pm \sqrt{1 + c_{mn}} e^{k_n x + w_n t} \right|_{N \times N} - 1 \right\}, \quad i = 1, 2. \quad (5.56)$$

where $|\cdot|_{N \times N}$ stands for determinant, $c_{mn} = \frac{s_m^- s_n^- - s_m^+ s_n^+}{(s_m^- - s_m^+)(s_n^- - s_n^+)} \left(\frac{\sqrt{s_m^+ s_n^-} - \sqrt{s_m^- s_n^+}}{\sqrt{s_m^- s_n^-} - \sqrt{s_m^+ s_n^+}} \right)^2$, and w_n, k_n are the same as in (5.54). Notice that $c_{mn} = c_{nm}$, $c_{nn} = 0$. We have omitted the index (q) in all the parameters above. The N-dark-dark soliton (5.55) possesses $2N+4$ real parameters, i.e. ρ_j^\pm, k_m, c_m . Let us mention that we have verified these solutions up to $N = 3$ using the MATHEMATICA program. The above solutions associated to the two (5.51)-(5.52) or higher order (5.55)-(5.56) dark-dark-solitons are in general non-degenerate. Notice that in the CNLS case the two- and higher-dark-dark solitons derived in [10] are actually degenerate and reducible to scalar NLS dark solitons. So, regarding this property our solutions resemble to the ones recently obtained in [4] for the r -CNLS system and in [14] for the AKNS $_r$ model.

Let us remark that the free field NVBC (3.29) requires the introduction of more parameters into the N-soliton tau functions (5.55)-(5.56) through the modified DT. So, following similar steps to get (5.49) one has: i) the parameters s_n^\pm entering the tau function τ_0 in (5.55) remain the same. ii) in (5.56) the factor ρ_j^\pm must be changed to $\rho_j^\pm e^{a_j^\pm x + b_j^\pm t}$. iii) the parameters s_n^\pm entering the tau functions τ_i^\pm in (5.56) must be changed as $s_n^\pm \rightarrow s_n^\pm \chi_i^\pm$. This general N -soliton will have, in addition to the $\Omega_{1,2}$ parameters, $3N + 8$ real parameters, i.e. $\rho_j^\pm, k_m, c_{m,j}, a_j^\pm$.

Similar constructions can be performed to get the singular and the mixed dark-singular N-solitons. In [4] the general non-degenerate N-dark-dark solitons in the r -CNLS model with defocusing and mixed nonlinearity have been reported in the context of the KP-hierarchy reduction approach, and in [14] the bright and dark multi-soliton solutions of the AKNS $_r$ system have been addressed in the algebro-geometric approach.

5.3 Dark-dark soliton bound states

So far, reports on multi-dark-dark soliton bound states in integrable systems, to our knowledge, are very limited. Recently, it has been shown that in the mixed-nonlinearity case of the 2-CNLS system, two dark-dark solitons can form a stationary bound state [4]. Then, in order to have multi-dark-dark soliton bound states in the N-soliton solution (5.55)-(5.56) the constituent solitons should have the same velocity, i.e. denoting $y_n \equiv \frac{s_n^+}{s_n^-}$, ($y_n \notin [0, 1]$) then $\frac{w_n}{k_n} = \frac{y_n + 1}{y_n - 1} k_n \equiv v$ (assume $v > 0$) for certain soliton parameters labeled by $n = 1, 2, \dots$. We show that the signs of the sum $\sum_j \rho_j^+ \rho_j^-$ determine the existence of these bound states, for the positive sign bound states can be formed, whereas for the negative sign they do not exist.

First, consider $\sum_i \rho_i^+ \rho_i^- > 0$, then from (5.54) it follows that $k_n = \frac{|y_n| - 1}{\sqrt{|y_n|}} (\sum_i \rho_i^+ \rho_i^-)^{1/2}$ for $y_n < 0$. Therefore, in order for multi-dark-dark soliton bound states to exist the equation

$$\frac{(|y_n| - 1)^2}{(|y_n| + 1)\sqrt{|y_n|}} = \frac{v}{\sqrt{\sum_i \rho_i^+ \rho_i^-}} \equiv c > 0 \quad (5.57)$$

must give at least two distinct positive solutions for $|y_n|$. In fact, one has the following two solutions

$$|y_{1,2}| = 1 + \frac{c^2}{4} + c \frac{\sqrt{16+c^2}}{4} \pm \frac{\sqrt{12c^2+c^4+\Delta}}{2\sqrt{2}}, \quad (5.58)$$

where $\Delta \equiv \frac{64c+20c^3+c^5}{\sqrt{16+c^2}}$. These exhaust all the possible solutions with the condition (5.57). The right hand side of (5.58) with either $+$ or $-$ signs provides $|y_{1,2}| > 0$ for any $c > 0$. The other possible solutions $y_{3,4}$ do not satisfy the above requirements, since they possess a term of the form $\sqrt{12c^2+c^4-\Delta}$, which is imaginary for any positive value c . Therefore, two-dark-dark-soliton bound states exist in the $sl(3)$ -AKNS system, and three- and higher-dark-dark-soliton bound states can not exist. These results hold for any value of the index q . Notice that in order to reduce to the 2-CNLS system the parameter relationship (5.41) must be satisfied. So, from (5.41) one has $\sum_i \rho_i^+ \rho_i^- = \frac{1}{\mu} [\frac{|\rho_1^+|^2}{\delta_1} + \frac{|\rho_2^+|^2}{\delta_2}] < 0$ (remember that under $\rho_i^\pm \rightarrow i\rho_i^\pm$ the sum $\sum_i \rho_i^+ \rho_i^-$ reverses sign); so, it is possible if $\delta_1 = -\delta_2$ (e.g. $\delta_1 = -\delta_2 = -1$ for $|\rho_1^+| > |\rho_2^+|$), this case corresponds to the 2-CNLS with mixed focusing and defocusing nonlinearities. Thus, the two-dark-dark soliton bound state solution we have obtained here corresponds to the Manakov model with mixed nonlinearity. The case $\delta_i = -1$ ($i = 1, 2$) defines the defocusing Manakov model which does not support multi-dark-dark-soliton bound states [4].

Second, the condition $\sum_i \rho_i^+ \rho_i^- < 0$ will provide only a single positive solution for $y^q > 0$ in the equation $\frac{w_n}{k_n} = v > 0$. So, one can not obtain two or more solitons with the same velocity and therefore bound states in this case are not possible.

6 Mixed boundary conditions and dark-bright solitons

One may ask about the mixed boundary conditions for the system (2.6)-(2.7), i.e. NVBC for one of the field components of the system, say Ψ_1^\pm , and VBC for the other field component, Ψ_2^\pm . So, we will deal with the mixed boundary condition (3.4). Moreover, taking into account sequence of conditions as in (2.10)

$$t \rightarrow -it, \quad [\Psi_i^+]^* = -\mu \delta_i \Psi_i^- \equiv -\mu \delta_i \psi_i, \quad (6.1)$$

the system (2.6)-(2.7), for $(\beta_i \cdot \vec{\rho})$ being real, can be written as

$$i \partial_t \psi_k + \partial_x^2 \psi_k + 2\mu \left[\sum_{j=1}^2 \delta_j |\psi_j|^2 - \frac{1}{2} (\beta_k \cdot \vec{\rho}) \right] \psi_k = 0, \quad k = 1, 2. \quad (6.2)$$

Precisely, the system (6.2) in the defocusing case ($\delta_1 = \delta_2 = -1$) and possessing the first trivial solution (3.4), i.e. $\rho_1 \neq 0$; $\rho_2^\pm = 0$, has been considered in [26] in order to investigate dark-bright solitons which describe an inhomogeneous two-species Bose-Einstein condensate. The system (2.6)-(2.7) with the mixed trivial solution (3.4) can be written as

$$\partial_t \Psi_1^\pm = \pm \partial_x^2 \Psi_1^\pm \mp 2 \left[\sum_{j=1}^2 \Psi_j^+ \Psi_j^- - \rho_1^+ \rho_1^- \right] \Psi_1^\pm, \quad (6.3)$$

$$\partial_t \Psi_2^\pm = \pm \partial_x^2 \Psi_2^\pm \mp 2 \left[\sum_{j=1}^2 \Psi_j^+ \Psi_j^- - \frac{1}{2} \rho_1^+ \rho_1^- - \frac{3}{2} \Omega_2 \right] \Psi_2^\pm. \quad (6.4)$$

From the previous sections we can make the following observation: the vacuum connections relevant to each type of solutions must exhibit the fact that dark solitons are closely related to NVBC, whereas bright solitons to the relevant VBC one. Since Ω_2 and Ω_1 are some parameters satisfying $2\Omega_1 + \Omega_2 = 2\rho_1^+ \rho_1^-$, for simplicity we will assume $\Omega_2 = 0$, $\Omega_1 = \rho_1^+ \rho_1^-$ in (3.6)-(3.7). Therefore, the connections (3.6)-(3.7) for the mixed (constant-zero) boundary condition (3.4) take the form

$$\hat{A}^{vac} \equiv \tilde{\varepsilon}_{\beta_1}^1 = E^{(1)} + \rho_1^+ E_{\beta_1}^{(0)} + \rho_1^- E_{-\beta_1}^{(0)}, \quad (6.5)$$

$$\hat{B}^{vac} \equiv \tilde{\varepsilon}_{\beta_1}^2 = E^{(2)} + \rho_1^+ E_{\beta_1}^{(1)} + \rho_1^- E_{-\beta_1}^{(1)} \quad (6.6)$$

The relevant group element $\Psi_{mbc}^{(0)}$ is given by

$$\Psi_{mbc}^{(0)} \equiv e^{x \tilde{\varepsilon}_{\beta_1}^1 + t \tilde{\varepsilon}_{\beta_1}^2}, \quad (6.7)$$

In order to construct the soliton solutions we must look for the common eigenstates of the adjoint action of the vacuum connections (6.5)-(6.6). So, one has

$$[\tilde{\varepsilon}_{\beta_1}^1, \Gamma_{\pm\beta_2}] = \pm k^\pm \Gamma_{\pm\beta_2}, \quad (6.8)$$

$$[\tilde{\varepsilon}_{\beta_1}^2, \Gamma_{\pm\beta_2}] = \pm w^\pm \Gamma_{\pm\beta_2}, \quad w^\pm = (k^\pm)^2 - \rho_1^+ \rho_1^- \quad (6.9)$$

where the vertex operators $\Gamma_{\pm\beta_2}(k^\pm, \rho_1^\pm)$ are defined in (C.3). We expect that these vertex operators will be associated to the bright-dark soliton solutions of the model.

Let us write the following expressions

$$[x \tilde{\varepsilon}_{\beta_1}^1 + t \tilde{\varepsilon}_{\beta_1}^2, \Gamma_{\beta_2}] = \varphi^+(x, t) \Gamma_{\beta_2}; \quad \varphi^+(x, t) = k^+ x + w^+ t; \quad (6.10)$$

$$[x \tilde{\varepsilon}_{\beta_1}^1 + t \tilde{\varepsilon}_{\beta_1}^2, \Gamma_{-\beta_2}] = -\varphi^-(x, t) \Gamma_{-\beta_2}; \quad \varphi^-(x, t) = k^- x + w^- t. \quad (6.11)$$

The 1-dark-bright soliton is constructed taking h in (3.26)-(3.28) as

$$h = e^{\gamma^+ \Gamma_{+\beta_2}} e^{\gamma^- \Gamma_{-\beta_2}}, \quad \gamma^\pm = \text{constants}, \quad (6.12)$$

where the vertex operators in (C.3) have been considered. Notice that due to the nilpotency property of the vertex operators, as presented in the appendix C, the exponential series must truncate. So, replacing the group element (6.12) in (3.26)-(3.28) one has the following tau functions ³

$$\bar{\tau}_1^\pm = e^{\varphi^+ - \varphi^-} \gamma^+ \gamma^- < \lambda_o | E_{\mp\beta_1}^{(1)} \Gamma_{+\beta_2} \Gamma_{-\beta_2} | \lambda_o > \quad (6.13)$$

$$\bar{\tau}_2^\pm = e^{\pm \varphi^\pm} \gamma^\pm < \lambda_o | E_{\mp\beta_2}^{(1)} \Gamma_{\pm\beta_2} | \lambda_o > \quad (6.14)$$

$$\bar{\tau}_0 = 1 + e^{\varphi^+ - \varphi^-} \gamma^+ \gamma^- < \lambda_o | \Gamma_{+\beta_2} \Gamma_{-\beta_2} | \lambda_o > \quad (6.15)$$

³There exist other eigenstates $[\tilde{\varepsilon}_{\beta_1}^1, V_{\beta_1}^q] = \lambda V_{\beta_1}^q$; $[\tilde{\varepsilon}_{\beta_1}^2, V_{\beta_1}^q] = (-1)^{q-1} \lambda (\lambda^2 - \rho_0^2)^{1/2} V_{\beta_1}^q$, where $\rho_0^2 = 4\rho_1^+ \rho_1^-$ and $V_{\beta_1}^q$ ($q = 1, 2$) is given in (C.5). However, these eigenstates are related to purely dark-solitons for the first component Ψ_1^\pm , e.g. if one takes the group element $h = e^{V_{\beta_1}^q}$, it will not excite the second component Ψ_2^\pm since the matrix elements of type $< \lambda_o | E_{\mp\beta_2}^{(1)} V_{\beta_1}^q | \lambda_o >$ vanish.

These tau functions resemble the ones in (5.36)-(5.37) for the 1-dark soliton (5.39) and the eqs. (4.7)-(4.8) for the 1-bright soliton (4.9) components, respectively. The matrix elements in (6.13)-(6.15) can be computed, and will depend only on the parameters k^\pm, ρ_1^\pm . So, one has six independent parameters $\gamma^\pm, k^\pm, \rho_1^\pm$ associated to these tau functions. Let us write the tau functions in the form $\bar{\tau}_0 = 1 + c_0 e^{k^+ x + w^+ t} e^{-k^- x - w^- t}$, $\bar{\tau}_1^\pm = a^\pm e^{k^+ x + w^+ t} e^{-k^- x - w^- t}$, $\bar{\tau}_2^\pm = b^\pm e^{\pm k^\pm x \pm w^\pm t}$, where $w^\pm = (k^\pm)^2 - \rho_1^+ \rho_1^-$; $c_0 = \frac{a^+ a^-}{a^+ \rho_1^- - a^- \rho_1^+}$; $\frac{a^+}{a^-} = \frac{k^+ \rho_1^+}{k^- \rho_1^-}$; $b^+ b^- = (\frac{a^-}{\rho_1^-})(\frac{k^+}{k^-} - 1)(k^+ k^- + \rho_1^+ \rho_1^-)$. So, using these tau functions in (3.25) one has

$$\Psi_1^\pm = \rho_1^\pm \pm \left(\frac{a^\pm}{2c_0}\right) \left[1 + \tanh \frac{(k^+ - k^-)x + (w^+ - w^-)t}{2}\right] \quad (6.16)$$

$$\Psi_2^\pm = \frac{b^\pm}{2\sqrt{c_0}} e^{\pm \frac{1}{2}[(k^+ + k^-)x + (w^+ + w^-)t]} \operatorname{sech} \frac{(k^+ - k^-)x + (w^+ - w^-)t}{2}. \quad (6.17)$$

This is the 1-dark-bright soliton of the $\hat{sl}(3)$ AKNS model (6.3)-(6.4) (for $\Omega_2 = 0$). Notice that this solution has six independent real parameters, say $a^-, b^-, k^\pm, \rho_1^\pm$.

The construction of the 2-dark-bright solitons follows similar steps. The group element

$$h = e^{\gamma_1^+ \Gamma + \beta_2(k_1^+)} e^{\gamma_1^- \Gamma - \beta_2(k_1^-)} e^{\gamma_2^+ \Gamma + \beta_2(k_2^+)} e^{\gamma_2^- \Gamma - \beta_2(k_2^-)} \quad (6.18)$$

does the job. We record the relevant tau functions

$$\bar{\tau}_0 = 1 + \sum_{m,n=1}^2 c_{mn} e^{k_m^+ x + w_m^+ t} e^{-k_n^- x - w_n^- t} + c_0 e^{k_1^+ x + w_1^+ t} e^{-k_1^- x - w_1^- t} e^{k_2^+ x + w_2^+ t} e^{-k_2^- x - w_2^- t} \quad (6.19)$$

$$\bar{\tau}_1^\pm = \sum_{m,n=1}^2 a_{mn}^\pm e^{k_m^+ x + w_m^+ t} e^{-k_n^- x - w_n^- t} + d_1^\pm e^{k_1^+ x + w_1^+ t} e^{-k_1^- x - w_1^- t} e^{k_2^+ x + w_2^+ t} e^{-k_2^- x - w_2^- t} \quad (6.20)$$

$$\bar{\tau}_2^\pm = \sum_{m=1}^2 b_m^\pm e^{\pm k_m^\pm x \pm w_m^\pm t} + e^{\pm k_1^\pm x \pm w_1^\pm t} e^{\pm k_2^\pm x \pm w_2^\pm t} \left[\sum_{m=1}^2 q_m^\pm e^{\mp k_m^\mp x \mp w_m^\mp t} \right], \quad (6.21)$$

where $w_n^\pm = (k_n^\pm)^2 - \rho_1^+ \rho_1^-$; $c_{mn} = \frac{a_{mn}^+ a_{mn}^-}{a_{mn}^+ \rho_1^- - a_{mn}^- \rho_1^+}$; $\frac{a_{mn}^+}{a_{mn}^-} = \frac{k_m^+ \rho_1^+}{k_n^- \rho_1^-}$; $c_0 = \frac{d_1^+ d_1^-}{d_1^+ \rho_1^- - d_1^- \rho_1^+}$; $\frac{d_1^+}{d_1^-} = \frac{k_1^+ k_2^+ \rho_1^+}{k_1^- k_2^- \rho_1^-}$; $\frac{a_{12}^-(k_1^+ - k_2^-)(k_1^+ k_2^- + \rho_1^+ \rho_1^-)}{a_{11}^-(k_1^+ - k_1^-)(k_1^+ k_1^- + \rho_1^+ \rho_1^-)} + \frac{a_{22}^-(k_2^+ - k_2^-)(k_2^+ k_2^- + \rho_1^+ \rho_1^-)}{a_{21}^-(k_1^+ - k_2^-)(k_1^+ k_2^- + \rho_1^+ \rho_1^-)} = 0$; $b_i^+ b_i^- = \frac{(k_i^+ - k_i^-)(k_i^+ k_i^- + \rho_1^+ \rho_1^-)}{k_i^- \rho_1^-} a_{ii}^-$; $d_1^- = -\frac{a_{12}^+ a_{21}^-(k_1^+ - k_2^-)^2 (k_1^- - k_2^-) (k_1^+ k_2^+ - k_1^- k_2^-) (k_1^+ k_2^+ + \rho_1^+ \rho_1^-) (k_1^- k_2^- + \rho_1^- \rho_1^-)}{(k_1^+ - k_1^-)^2 (k_1^- - k_2^-) (k_1^+ - k_2^-) (k_2^+ - k_2^-)^2 \rho_1^- (k_1^+ k_1^- + \rho_1^+ \rho_1^-) (k_2^+ k_2^- + \rho_1^+ \rho_1^-)}$; $\frac{b_2^-}{b_1^-} = \frac{a_{12}^+ k_1^- (k_1^+ - k_2^-) (k_1^+ k_2^- + \rho_1^+ \rho_1^-)}{a_{11}^+ k_2^- (k_1^+ - k_1^-) (k_1^+ k_1^- + \rho_1^+ \rho_1^-)}$; $q_i^\pm = \mp \frac{a_{ii}^\pm (k_i^\mp)^2}{b_i^\pm k_1^\pm k_2^\pm (\rho_1^\pm)^2} \frac{1}{(k_i^+ - k_i^-)^2} \frac{(k_i^\pm - k_2^\pm)^2}{(k_i^\pm - k_{3-i}^\pm)} (k_1^\pm k_2^\pm + \rho_1^\pm \rho_1^\mp) a_{m_i^\pm n_i^\pm}^\pm$; $m_i^+ = n_i^- = 3 - i$; $m_i^- = n_i^+ = i$.

Notice that this solution possesses 10 independent real parameters, namely, $a_{mm}^-, a_{12}^-, b_1^-, k_m^\pm, \rho_1^\pm$, ($m = 1, 2$). These tau functions resemble to the ones in [24] provided for the 2-CNLS model. The generalization to N-dark-bright solitons requires the group element h to be

$$h = e^{\gamma_1^+ \Gamma + \beta_2(k_1^+)} e^{\gamma_1^- \Gamma - \beta_2(k_1^-)} e^{\gamma_2^+ \Gamma + \beta_2(k_2^+)} e^{\gamma_2^- \Gamma - \beta_2(k_2^-)} \dots e^{\gamma_N^+ \Gamma + \beta_2(k_N^+)} e^{\gamma_N^- \Gamma - \beta_2(k_N^-)}. \quad (6.22)$$

7 Generalization to AKNS_r ($r \geq 3$) model

The procedures presented so far can directly be extended to the AKNS_r ($r \geq 3$) model for the affine Kac-Moody algebra $\hat{sl}(n)$ furnished with the homogeneous gradation. According to the construction in [20], in this case the equations of motion will describe the dynamics of the fields Ψ_j^\pm ($j = 1, 2, \dots, r$; $r \equiv n - 1$) associated to the generators $E_{\pm\beta_j}^{(0)}$, where the β_j are the positive roots defined by $\beta_j \equiv \alpha_j + \alpha_{j+1} \dots + \alpha_r$ (α_j = simple roots). The outcome will be the eqs. in (2.6)-(2.7) with $2r$ real fields.

The DT methods would be applied following similar steps as in section 3 and subsection 3.1, for constant and free field NVBC's, respectively. In particular, in the constant NVBC the form of the relationships (3.25) and (3.26)-(3.28) will remain the same, except that $i = 1, 2, \dots, r$. In the case of free field NVBC the relationships (5.47)-(5.48) would be satisfied with $i = 1, 2, \dots, r$.

The VBC and the bright solitons will be associated to the vertex operators F_j, G_j (see (C.1)) as in section 4. In this case, one requires $\rho_i^\pm = 0$, ($i = 1, 2, \dots, r$) in (3.25). The dark solitons, as in section 5, will require the vertex operator of type $W^q(k, \rho_j^\pm)$ ($j = 1, 2, \dots, r$), the analog of the operator in (C.7) incorporating additional terms. Finally, the mixed boundary conditions and the dark-bright solitons will emerge by extending the discussion in section (6). In the case of the vector 1-soliton solution it is possible to form the combination $(m, r - m)$, m = number of dark components, $r - m$ = number of bright components. So, the vertex operators analog to $\Gamma_{\pm\beta_2}(k^\pm, \rho_1^\pm)$ in (C.3) will be associated to the roots $\pm\beta_j$, $(\pm\beta_j \mp \beta_i)$ ($i = 1, 2, \dots, m$) such that $\Gamma_{\pm\beta_j}(k^\pm, \rho_i^\pm)$ ($j = m + 1, m + 2, \dots, r$) .

8 Discussion

We have considered soliton type solutions of the AKNS model supported by the various boundary conditions (3.2)-(3.5): vanishing, (constant) non-vanishing and mixed vanishing-nonvanishing boundary conditions related to bright, dark and bright-dark soliton solutions, respectively, by applying the DT approach as presented in [15]. The set of solutions of the AKNS_r system (2.6)-(2.7) is much larger than the solutions of the r-CNLS system (2.11). A subset of solutions of the AKNS_r system, (2.6)-(2.7) for $r = 2$ and (5.4)-(5.5) for $r = 1$, respectively, solve the scalar NLS (5.7) and 2-CNLS system (2.11), under relevant complexifications.

Moreover, the free field boundary condition (3.29) for dark solitons is considered in the context of a modified DT approach associated to the dressing group [29], and the general N-dark-dark soliton solutions of the AKNS₂ system have been derived. These soliton components are not proportional to each other and thus they do not reduce to the AKNS₁ solitons, in this sense they are not degenerate. We showed that these solitons under convenient complexifications reduce to the general N-dark-dark solitons derived previously in the literature for the CNLS model [4, 14]. In addition, we have shown that two-dark-dark-soliton bound states exist in the $sl(3)$ -AKNS system, and three- and higher-dark-dark-soliton bound states can not exist. These results hold for any value of the index q . In the case of reduced 2-CNLS when focusing and

defocusing nonlinearities are mixed, this result corresponds to 2-dark-dark soliton stationary bound state [4].

In the mixed constant boundary conditions we derived the dark-bright solitons of the $\hat{sl}(3)$ AKNS model. These solitons under the complexification (6.1) reduce to the solitons of the 2-CNLS model (6.2) which will be useful in order to investigate dark-bright solitons appearing in an inhomogeneous two-species Bose-Einstein condensate [26].

The relevant steps toward the $AKNS_r$ ($r \geq 3$) extension were briefly discussed in the framework of the DT methods. In particular, the vertex operator calculations can be extended in a direct way following the same steps as in the appendices B and C and the $\hat{sl}(n)$ highest weight representation [42].

Another point we should highlight relies upon the possible relevance of the CNLS tau functions to its higher-order generalizations. We expect that the tau functions of the higher-order CNLS generalization are related somehow to the basic tau functions of the usual CNLS equations. This fact is observed for example in the case of the coupled scalar NLS+ derivative-NLS system in which the coupled system possesses a composed tau function depending on the basic scalar NLS tau functions [41].

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A $\hat{sl}(2)$ matrix elements

The commutation relations for the $\hat{sl}(2)$ affine KacMoody algebra elements are

$$[H^{(m)}, H^{(n)}] = 2m\delta_{m+n,0}C, \quad (A.1)$$

$$[H^{(m)}, E_{\pm}^{(n)}] = \pm 2E_{\pm}^{(m+n)}, \quad (A.2)$$

$$[E_+^{(m)}, E_-^{(n)}] = H^{(m+n)} + m\delta_{m+n,0}C; \quad (A.3)$$

$$[D, T_a^{(m)}] = mT_a^{(m)}; \quad T_a^{(m)} = \{H^{(m)}, E_{\pm}^{(m)}\} \quad (A.4)$$

The central extension ensures highest weight representations (h.w.r.) of the affine algebra (see e.g. [15]). So, in the h.w.r. $\{|\lambda_0\rangle, |\lambda_1\rangle\}$ one has the following relationships

$$E_+^{(0)}|\lambda_a\rangle = 0 \quad (A.5)$$

$$E_{\pm}^{(m)}|\lambda_a\rangle = 0, \quad m > 0 \quad (A.6)$$

$$H^{(m)}|\lambda_a\rangle = 0, \quad m > 0; \quad (A.7)$$

$$H^{(0)}|\lambda_a\rangle = \delta_{a,1}|\lambda_a\rangle, \quad (A.8)$$

$$C|\lambda_a\rangle = |\lambda_a\rangle \quad (A.9)$$

where $a = 0, 1$. The adjoint relations $(E_{\pm}^{(m)})^{\dagger} = E_{\mp}^{(-m)}$, $(H^{(m)})^{\dagger} = H^{(-m)}$ allow one to know their actions on the $|\lambda_a\rangle$. Next, consider the vertex operators

$$\begin{aligned} \hat{V}^q(\gamma, \hat{\rho}) = & \sum_{n=-\infty}^{\infty} \{(\gamma^2 - \hat{\rho}^2)^{-n/2} [e_q]^n \left[H^{(n)} - \frac{\hat{\rho}^+}{\gamma - e_q (\gamma^2 - \hat{\rho}^2)^{1/2}} E_+^{(n)} + \right. \\ & \left. \frac{\hat{\rho}^-}{\gamma + e_q (\gamma^2 - \hat{\rho}^2)^{1/2}} E_-^{(n)} \right] + e_q \left(\frac{\gamma^2 - \hat{\rho}^2}{\gamma^2} \right)^{1/2} \delta_{n,0} C\}; \quad q = 1, 2; \end{aligned} \quad (\text{A.10})$$

where $e_q \equiv (-1)^{q-1}$ and $\hat{\rho}^2 \equiv \hat{\rho}^+ \hat{\rho}^-$. The vertex operator $\hat{V}^q(\gamma, \hat{\rho})$ satisfies

$$[\hat{\varepsilon}_1, \hat{V}^q] = 2\gamma \hat{V}^q, \quad [\hat{\varepsilon}_2, \hat{V}^q] = (-1)^{q-1} 2\gamma (\gamma^2 - \hat{\rho}^2)^{1/2} \hat{V}^q, \quad q = 1, 2, \quad (\text{A.11})$$

where

$$\hat{\varepsilon}_1 = H^{(1)} + \hat{\rho}^+ E_+^{(0)} + \hat{\rho}^- E_-^{(0)}, \quad \hat{\varepsilon}_2 = H^{(2)} + \hat{\rho}^+ E_+^{(1)} + \hat{\rho}^- E_-^{(1)}. \quad (\text{A.12})$$

The following matrix elements can be computed using the properties (A.5)-(A.9)

$$\langle \lambda_o | \hat{V}^q | \lambda_o \rangle, \quad = \quad e_q \frac{(\gamma^2 - \hat{\rho}^2)^{1/2}}{\gamma}, \quad (\text{A.13})$$

$$\langle \lambda_o | E_{\mp}^{(1)} \hat{V}^q | \lambda_o \rangle = \mp \frac{2\hat{\rho}^{\pm}}{\gamma \mp e_q (\gamma^2 - \hat{\rho}^2)^{1/2}} (\gamma^2 - \hat{\rho}^2)^{1/2}. \quad (\text{A.14})$$

The matrix element $\langle \lambda_o | \hat{V}^q(\gamma_1, \hat{\rho}) \hat{V}^q(\gamma_2, \hat{\rho}) | \lambda_o \rangle$ can be computed by developing the products and keeping only non-trivial terms, then one makes use of the commutation rules to change the order, and eventually to get some central terms C . The double sum can be simplified to a single sum and each term can be substituted by power series like $\sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)}$, $\sum_{n=1}^{\infty} x^n = \frac{x}{(1-x)}$, and $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$. So, one can get

$$\begin{aligned} \langle \lambda_o | \hat{V}^q(\gamma_1, \hat{\rho}) \hat{V}^q(\gamma_2, \hat{\rho}) | \lambda_o \rangle = & [2 + 2K(\gamma_1, \gamma_2)] \frac{K_0(\gamma_1, \gamma_2)}{[1 - K_0(\gamma_1, \gamma_2)]^2} + \\ & \left[\frac{K(\gamma_1, \gamma_2) \hat{\rho}^2}{\gamma_1 \gamma_2} + 1 \right]; \quad q = 1, 2 \end{aligned} \quad (\text{A.15})$$

$$\text{where } K_0(\gamma_1, \gamma_2) \equiv \frac{\sqrt{\gamma_2^2 - \hat{\rho}^2}}{\sqrt{\gamma_1^2 - \hat{\rho}^2}}; \quad K(\gamma_1, \gamma_2) \equiv \frac{\sqrt{\gamma_1^2 - \hat{\rho}^2} \sqrt{\gamma_2^2 - \hat{\rho}^2} - \gamma_1 \gamma_2}{\hat{\rho}^2} \quad (\text{A.16})$$

In order to prove the nilpotency property of the vertex operator $V^q(\gamma_1, \hat{\rho})$, when evaluated within the state $|\lambda_o\rangle$, it is convenient to write (A.15) in the following Laurent series expansion

$$\begin{aligned} \langle \lambda_o | \hat{V}^q(\gamma_1, \hat{\rho}) \hat{V}^q(\gamma_2, \hat{\rho}) | \lambda_o \rangle = & -6\hat{\rho}^2 \sqrt{\gamma_1^2 - \hat{\rho}^2} \sqrt{\gamma_2^2 - \hat{\rho}^2} \left(\frac{\sqrt{\gamma_1^2 - \hat{\rho}^2} + \sqrt{\gamma_2^2 - \hat{\rho}^2}}{\gamma_2^2 - \hat{\rho}^2} \right)^2 \left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \right)^2 \times \\ & \left[\frac{1}{4!} - \frac{10\gamma_2}{(\gamma_2^2 - \hat{\rho}^2)} \frac{(\gamma_1 - \gamma_2)}{5!} + \frac{15(6\gamma_2^2 + \hat{\rho}^2)}{(\gamma_2^2 - \hat{\rho}^2)^2} \frac{(\gamma_1 - \gamma_2)^2}{6!} - \right. \\ & \left. \frac{420\gamma_2(2\gamma_2^2 + \hat{\rho}^2)}{(\gamma_2^2 - \hat{\rho}^2)^3} \frac{(\gamma_1 - \gamma_2)^3}{7!} + \dots \right]. \end{aligned} \quad (\text{A.17})$$

From this, it is clear that

$$\lim_{\gamma_1 \rightarrow \gamma_2} \langle \lambda_o | \hat{V}^q(\gamma_1, \hat{\rho}) \hat{V}^q(\gamma_2, \hat{\rho}) | \lambda_o \rangle \rightarrow 0, \quad q = 1, 2. \quad (\text{A.18})$$

B The affine Kac-Moody algebra $\widehat{sl}_3(C)$

In the following we provide some results about the affine Kac-Moody algebra $\mathcal{G} = \widehat{sl}_3(C)$ relevant to our discussions above. We follow closely [42, 18]. The elements of the $sl_3(C)$ Lie algebra are all 3×3 complex matrices with zero trace. Consider the corresponding root system $\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$, such that the three positive roots are α_i , $i = 1, 2, 3$, with α_a , $a = 1, 2$, being the simple roots and $\alpha_3 = \alpha_1 + \alpha_2$. We choose a standard basis for the Cartan subalgebra \mathcal{H} such that

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (\text{B.1})$$

and the generators of the root subspaces corresponding to the positive roots are chosen as

$$E_{+\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{+\alpha_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{B.2})$$

For negative roots one has $E_{-\alpha} = (E_{+\alpha})^T$. The invariant bilinear form on $sl_3(C)$, $(x|y) = \text{tr}(xy)$; $x, y \in sl_3(C)$ induces a nondegenerate bilinear form on \mathcal{H}^* which we also denote by $(\cdot|\cdot)$. This definition allows one to write

$$(\alpha_1|\alpha_1) = 2, \quad (\alpha_2|\alpha_2) = 2, \quad (\alpha_1|\alpha_2) = -1. \quad (\text{B.3})$$

On the other hand, the generators $T^{(m)} \equiv \{H_1^{(m)}, H_2^{(m)}, E_{\alpha}^{(m)}\}$, where $m \in \mathbb{Z}$ and $\alpha \in \Delta$, together with the central C and the 'derivation' operator D ($[D, T^{(m)}] = mT^{(m)}$) form a basis for $\widehat{sl}_3(C)$. These generators satisfy the commutation relations

$$[H_a^{(m)}, H_b^{(n)}] = m(\alpha_a|\alpha_b) C \delta_{m+n,0}, \quad (\text{B.4})$$

$$[H_a^{(m)}, E_{\pm\alpha_i}^{(n)}] = \pm(\alpha_a|\alpha_i) E_{\pm\alpha_i}^{(m+n)}, \quad (\text{B.5})$$

$$[E_{\alpha_a}^{(m)}, E_{-\alpha_a}^{(n)}] = H_a^{(m+n)} + m C \delta_{m+n,0}, \quad (\text{B.6})$$

$$[E_{\alpha_3}^{(m)}, E_{-\alpha_3}^{(n)}] = H_1^{(m+n)} + H_2^{(m+n)} + m C \delta_{m+n,0}, \quad (\text{B.7})$$

$$[E_{\alpha_1}^{(m)}, E_{\alpha_2}^{(n)}] = E_{\alpha_3}^{(m+n)}, \quad (\text{B.8})$$

$$[E_{\alpha_3}^{(m)}, E_{-\alpha_1}^{(n)}] = -E_{\alpha_2}^{(m+n)}, \quad (\text{B.9})$$

$$[E_{\alpha_3}^{(m)}, E_{-\alpha_2}^{(n)}] = E_{\alpha_1}^{(m+n)}, \quad (\text{B.10})$$

$$[D, C] = 0, \quad (\text{B.11})$$

where $a, b = 1, 2$, $i = 1, 2, 3$ and $m, n \in \mathbb{Z}$. The remaining non-vanishing commutation relations are obtained by using the relation $[E_{\alpha}^{(m)}, E_{\beta}^{(n)}]^{\dagger} = -[E_{-\alpha}^{(-m)}, E_{-\beta}^{(-n)}]$.

In this paper we use the homogeneous \mathbb{Z} -gradation of $\hat{sl}_3(C)$ which is defined by the grading operator D , such that

$$\hat{sl}_3(C) = \bigoplus_{m \in \mathbb{Z}} \mathcal{G}_m, \quad [\mathcal{G}_m, \mathcal{G}_n] \subset \mathcal{G}_{m+n}, \quad (\text{B.12})$$

Where $\mathcal{G}_m = \{x \in \hat{sl}_3(C) \mid [D, x] = m x; m \in \mathbb{Z}\}$.

The subspace \mathcal{G}_0 is a subalgebra of $\hat{sl}_3(C)$ given by

$$\hat{\mathcal{G}}_0 = C H_1 \oplus C H_2 \oplus C E_\alpha^{(0)} \oplus C C \oplus C D \quad (\text{B.13})$$

and for the subspaces \mathcal{G}_m ($m \neq 0$) we have

$$\mathcal{G}_m = C H_1^{(m)} \oplus C H_2^{(m)} \oplus C E_{\alpha_1}^{(m)} \oplus C E_{\alpha_2}^{(m)} \oplus C E_{\alpha_3}^{(m)} \oplus C E_{-\alpha_1}^{(m)} \oplus C E_{-\alpha_2}^{(m)} \oplus C E_{-\alpha_3}^{(m)}. \quad (\text{B.14})$$

We use in the paper the fundamental highest weight representation $|\lambda_0\rangle$, satisfying

$$H_a^{(0)} |\lambda_0\rangle = 0, \quad E_\alpha^{(0)} |\lambda_0\rangle = 0, \quad C |\lambda_0\rangle = |\lambda_0\rangle \quad (\text{B.15})$$

for $a, b = 1, 2$, and $\alpha \in \Delta$. Such state is annihilated by all positive grade subspaces

$$\mathcal{G}_m |\lambda_0\rangle = 0, \quad m > 0, \quad (\text{B.16})$$

and all the representation space is spanned by the states obtained by acting on $|\lambda_0\rangle$ with negative grade generators. This representation space can be supplied with a scalar product such that one has

$$(H_a^m)^\dagger = H_a^{-m}, \quad (E_\alpha^m)^\dagger = E_{-\alpha}^{-m}, \quad (\text{B.17})$$

$$C^\dagger = C, \quad D^\dagger = D. \quad (\text{B.18})$$

It follows from (B.13) and (B.14) that

$$(\mathcal{G}_m)^\dagger = \mathcal{G}_{-m}, \quad (\text{B.19})$$

and therefore

$$\langle \lambda_0 | \mathcal{G}_{-m} = 0, \quad m > 0. \quad (\text{B.20})$$

In addition to the subalgebra \mathcal{g}_0 it is also convenient to consider two additional subalgebras

$$\mathcal{G}_{<0} = \bigoplus_{m>0} \mathcal{G}_m, \quad \mathcal{G}_{>0} = \bigoplus_{m>0} \mathcal{G}_{-m}. \quad (\text{B.21})$$

These subalgebras and the corresponding Lie groups play important role in the DT method.

The next relationships are useful in the AKNS_r ($r = 2$) model construction. The special element $E^{(l)}$ in the basis presented above can be written as

$$E^{(l)} = \frac{1}{3}(H_1^{(l)} + 2H_2^{(l)}), \quad [D, E^{(l)}] = lE^{(l)}. \quad (\text{B.22})$$

Then the matrix $E^{(0)}$ becomes

$$E^{(0)} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (\text{B.23})$$

The roots entering in the AKNS₂ construction are

$$\beta_1 \equiv \alpha_3 = \alpha_1 + \alpha_2; \quad \beta_2 = \alpha_2. \quad (\text{B.24})$$

Moreover, the following commutation relations hold

$$[E^{(l)}, H_a^{(m)}] = l \delta_{2a} C \delta_{l+m,0}, \quad a = 1, 2, \quad (\text{B.25})$$

$$[E^{(l)}, E_{\pm\beta_j}^{(m)}] = \pm E_{\pm\beta_j}^{(l+m)}, \quad j = 1, 2. \quad (\text{B.26})$$

$$[E^{(l)}, E_{\pm\beta_1 \mp \beta_2}^{(m)}] = 0, \quad (\text{B.27})$$

$$[H_1^{(m)}, E_{\pm\beta_1}^{(n)}] = \pm E_{\pm\beta_1}^{(m+n)}, \quad [H_1^{(m)}, E_{\pm\beta_2}^{(n)}] = \mp E_{\pm\beta_2}^{(m+n)} \quad (\text{B.28})$$

$$[H_2^{(m)}, E_{\pm\beta_1}^{(n)}] = \pm E_{\pm\beta_1}^{(m+n)}, \quad [H_2^{(m)}, E_{\pm\beta_2}^{(n)}] = \pm 2 E_{\pm\beta_2}^{(m+n)} \quad (\text{B.29})$$

$$[H_1^{(m)}, E_{\pm\beta_1 \mp \beta_2}^{(n)}] = \pm 2 E_{\pm\beta_1 \mp \beta_2}^{(m+n)} \quad (\text{B.30})$$

$$[H_2^{(m)}, E_{\pm\beta_1 \mp \beta_2}^{(n)}] = \mp E_{\pm\beta_1 \mp \beta_2}^{(m+n)} \quad (\text{B.31})$$

C $\hat{sl}(3)$ matrix elements

Consider the vertex operators associated to bright soliton solutions

$$F_j = \sum_{n=-\infty}^{+\infty} \nu_j^n E_{-\beta_j}^{(-n)}, \quad G_j = \sum_{n=-\infty}^{+\infty} \rho_j^n E_{\beta_j}^{(-n)}; \quad j = 1, 2; \quad \nu_j, \rho_j \in \mathbb{C}. \quad (\text{C.1})$$

It can be shown that they are nilpotent, i.e. $F_j^2 = 0, G_j^2 = 0$. The matrix element $\langle \lambda_o | F_j G_k | \lambda_o \rangle$ can be computed by developing the products and keeping only non-trivial terms, then one makes use of the commutation rules to get the central term C . The double sum can be simplified to a single sum, which provide the power series $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$. So, one has

$$\langle \lambda_o | F_j G_k | \lambda_o \rangle = \frac{\nu_j \rho_k}{(\nu_j - \rho_k)^2} \delta_{j,k} \quad (\text{C.2})$$

Let us consider the deformation of the vertex operators F_2, G_2 as

$$\Gamma_{\pm\beta_2}(k^{\pm}, \rho_1^{\pm}) = \sum_{n=-\infty}^{+\infty} \left(\frac{w^{\pm}}{k^{\pm}}\right)^{-n} [k^{\pm} E_{\pm\beta_2}^{(n)} - \rho_1^{\mp} E_{\pm\beta_2 \mp \beta_1}^{(n)}], \quad w^{\pm} = (k^{\pm})^2 - \rho_1^+ \rho_1^-. \quad (\text{C.3})$$

It is a direct computation to show the nilpotency of these operators, i.e. $\Gamma_{\pm\beta_2}^2 = 0$. Similar computations to the one in (C.2) provide the following matrix element

$$\langle \lambda_o | \Gamma_{\beta_2}(k^+) \Gamma_{-\beta_2}(k^-) | \lambda_o \rangle = \frac{w^+ w^- k^+ k^-}{(k^+ k^- + \rho_1^+ \rho_1^-)(k^+ - k^-)^2} \quad (\text{C.4})$$

Consider the vertex operator analog to the one in (A.10)

$$V_{\beta_1}^q(\lambda, \rho_0) = \sum_{n=-\infty}^{\infty} \{(\lambda^2 - \rho_0^2)^{-n/2} [e_q]^n \left[\frac{1}{2}(H_1^{(n)} + H_2^{(n)}) - \frac{\rho_1^+}{\lambda - e_q(\lambda^2 - \rho_0^2)^{1/2}} E_{\beta_1}^{(n)} + \frac{\rho_1^-}{\lambda + e_q(\lambda^2 - \rho_0^2)^{1/2}} E_{-\beta_1}^{(n)} \right] + e_q \frac{(\lambda^2 - \rho_0^2)^{1/2}}{2\lambda} \delta_{n,0} C\}; \quad q = 1, 2 \quad (\text{C.5})$$

where $e_q \equiv (-1)^{q-1}$ and $\rho_0^2 = 4\rho_1^+ \rho_1^-$. The next matrix element computation follows similar steps to the one performed to arrive at (A.15), except that one must take into account the $\hat{sl}(3)$ commutation rules. So, one has

$$\begin{aligned} \langle \lambda_o | V_{\beta_1}^q(\lambda_1, \rho_0) V_{\beta_1}^q(\lambda_2, \rho_0) | \lambda_o \rangle &= \frac{1}{4} \{ [2 + 2K(\lambda_1, \lambda_2)] \frac{K_0(\lambda_1, \lambda_2)}{[1 - K_0(\lambda_1, \lambda_2)]^2} + \\ &\quad [\frac{K(\lambda_1, \lambda_2) \rho_0^2}{\lambda_1 \lambda_2} + 1] \}; \quad q = 1, 2, \end{aligned} \quad (\text{C.6})$$

where K_0 and K are given in (A.16). Since this two-point function, except for an overall constant factor, is similar to the one in (A.15) one can use the relationships (A.17)-(A.18) to show that the operator $V_{\beta_1}^q(\lambda_1, \rho_0)$ is nilpotent.

The vertex operator generating the dark-dark soliton solution becomes

$$\begin{aligned} W^q(k, \rho_{1,2}^{\pm}) &= \sum_{n=-\infty}^{\infty} \{ (k^2 - 4 \sum_{i=1}^2 \rho_i^+ \rho_i^-)^{-n/2} [e_q]^n \left[s_1 H_1^{(n)} + s_2 H_2^{(n)} + \sum_{i=1}^2 e_{i(q)}^+ E_{\beta_i}^{(n)} + \right. \\ &\quad \left. \sum_{i=1}^2 e_{i(q)}^- E_{-\beta_i}^{(n)} + e_{12}^+ E_{\beta_1 - \beta_2}^{(n)} + e_{12}^- E_{\beta_2 - \beta_1}^{(n)} \right] + e_q \frac{(k^2 - 4 \sum_{i=1}^2 \rho_i^+ \rho_i^-)^{1/2}}{2k} \delta_{n,0} C \} \end{aligned} \quad (\text{C.7})$$

$$\begin{aligned} e_{i(q)}^{\pm} &= \frac{\mp \rho_i^{\pm}}{k \mp e_q (k^2 - 4 \sum_j \rho_j^+ \rho_j^-)^{1/2}}, \quad s_2 = \frac{1}{2}; \quad s_1 = \frac{1}{2} \frac{\rho_1^+ \rho_1^-}{\sum_i \rho_i^+ \rho_i^-}; \quad e_{12}^{\pm} = \frac{1}{2} \frac{\rho_1^{\pm} \rho_2^{\mp}}{\sum_i \rho_i^+ \rho_i^-}; \\ e_q &\equiv (-1)^{q-1} \end{aligned}$$

Notice that the vertex operator (C.7) reduces to the one in (C.5) in the limit $\rho_2^{\pm} \rightarrow 0$. The nilpotent property of this vertex operator can be verified as follows

$$\langle \lambda_o | W^1(k_1, \rho_{1,2}^{\pm}) W^1(k_2, \rho_{1,2}^{\pm}) | \lambda_o \rangle = \left(\frac{x_1 x_2}{4} \right) \frac{x_2 - x_1}{x_1^2 + S} \left(x_1 + \frac{1}{4} \frac{S - 4x_1^2}{x_1^2 + S} (x_2 - x_1) + \dots \right) \quad (\text{C.8})$$

where $x_1 = \sqrt{k_1^2 - 4S}$, $x_2 = \sqrt{k_2^2 - 4S}$, $S = \sum_j \rho_j^+ \rho_j^-$. In the limit $x_2 \rightarrow x_1$ (or $k_2 \rightarrow k_1$) the r. h. s. of eq. (C.8) vanishes.

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